Reversible-jump MCMC

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Suppose we have an collection of models $\bigcup_{k\in\neg} M_k$. Model M_i has a vector of parameters $\theta^{(i)}$ that belongs to R^{n_i}. The target is to sample $x \in \bigcup_{k\in\neg}(\{k\} \times R^{n_k})$. Thus, the parameter that will be updated should be $\{\theta^{(k)}, k\}$.

Dimension Matching

Assumption: $\int_A \pi(dx) \int_B q_m(x, dx')\alpha_m(x, x') = \int_A \pi(dx') \int_B q_m(x', dx)\alpha_m(x', x)$.

To make the assumption hold, $\alpha_m(x, x')$ was taken to be $min\{1, \frac{\pi(dx')q_m(x',dx)}{\pi(dx)q_m(x,dx')}\}$.

Switching between subspaces

First, suppose we have two subspaces $f_1 = \{1\} \times R^{n_1}$ and $f_2 = \{1\} \times R^{n_2}$ with density $p(\theta^{(1)}|k = 1)$ and $p(\theta^{(2)}|k = 2)$ separately (i.e. k = 2). Suppose the only move performed is to switch between two subspaces with probability $j(x)$. One way to swith between two subspaces is to generate a vector of continuous random variables $u^{(1)}$ with length m_1 such that $u^{(1)} \perp \theta^{(1)}$, then set $\theta^{(2)}$ be some deterministic function of $\theta^{(1)}$ and $u^{(1)}$. Reversely, we need to generate $u^{(1)}$ with length m_2 such that $u^{(2)} \perp \theta^{(2)}$ and set $\theta^{(1)}$ be some deterministic function of $\theta^{(2)}$ and $u^{(2)}$.

Requirement: there is a bijection between $(\theta^{(1)}, u^{(1)})$ and $(\theta^{(2)}, u^{(2)})$ $(n_1 + m_1 = n_2 + m_2)$.

Then, the proposal distribution q(x,dx') can be defined by prior distribution of $u^{(1)}$ and $u^{(2)}$.

Acceptance Probability : $\alpha(x, x') = min\{1, \frac{p(2,\theta^{(2)}|y)j(2,\theta^{(2)})q_2(u^{(2)})}{p(1,\theta^{(1)}|y)j(1,\theta^{(1)})q_2(u^{(1)})} \left|\frac{\partial(\theta^{(2)},u^{(2)})}{\partial(\theta^{(1)},u^{(1)})}\right|\}$

Notes:

1) m_1 or m_2 usually is taken to be zero in practice, thus no need to generate $u^{(1)}$ or $u^{(2)}$.

2) for more than two subspaces, there is no need to calculate all possible acceptance probability. Instead, only acceptance probability of adjacent subspaces need to be calculated (i.e. $\alpha_{f_1,f_2}, \alpha_{f_2,f_3}, ...$ Then, for example, the acceptance probability of switching from f_1 to f_3 is just $\alpha_{f_1,f_2} \times \alpha_{f_2,f_3}$.

Application: Possion change-point problem in one dimension

Prior distributions:

1) change-points (k) :

$k \sim possion(\lambda)$

Assume $k \in [1, k_{max}]$,

then there exist (k-1) change-points,

therefore $p(k) = e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$ \

2) positions (s) :

$\{s_2, ..., s_k\}$ generated from even-numbered order statistics from 2k+1 points $\sim U(s_1, s_{k+1})$

$\pi(s_2, ...s_k) = \frac{(2k-1)!}{(s_{k+1}-s_1)^{2k-1}} I_{s_1<s_2<...<s_k<s_{k+1}} \prod_{j=1}^{k}(s_{j+1} - s_j)$

3) heights (h) : $\{h_1, ..., h_k\} \sim \Gamma(\alpha, \beta)$

Four possible transitions:

1) change of height with probability η_k

2) change of position with probability π_k

3) birth of step with probability b_{k-1}

4) death of step with probability d_k

which satisfies:

1) $d_0 = \pi_0 = 0$

2) $b_{k_{max}} = 0$

3) $b_{k-1} = c\ min\{1, p(k)/p(k-1)\}$ and $d_k = c\ min\{1, p(k-1)/p(k)\}$, with constant c choosen as large as possible subject to $b_k + d_k \leq 0.9$.

4) $\eta_k = \pi_k$ for $k \neq 0$

Likelihood

Let m_j be the number of disaters which happened during $[s_j, s_{j+1})$; Let $x = (k, h_1, ..., h_k, s_2, ..., s_k)$ then, $P(y_1, ..., y_n|x) = exp(\sum_{j=1}^{k} m_j log(h_j) - \sum_{j=1}^{k}(s_{j+1-s_j})h_j)$

Proof

$$P((y_1, ..., y_n|x) = exp(\sum_{i=1}^{n} log(\lambda(y_i)) - \int_{s_1}^{s_{k+1}} \lambda(t)dt)$$

$$= exp(\sum_{j=1}^{k} m_j \times log(h_j) - \sum_{j=1}^{k}(s_{j+1} - s_j)h_j)$$

Transitions:

1. Height

Algorithm:

1) generate j randomly from 1 to k and u ∼ Uniform(0,1).
2) if accepted ($U < \alpha(h_j, h'_j)$), then update h_j to h'_j such that $log(\frac{h'_j}{h_j}) \sim U(-\frac{1}{2}, \frac{1}{2})$. Otherwise, $h'_j = h_j$

Proposal : $q(h_j, h'_j) = \frac{1}{h'_j}$

Acceptance probability : $\alpha(h_j, h'_j) = exp\{(m_j + \alpha - 2)(log(h'_j) - log(h_j)) - (s_{j+1} - s_j + \beta)(h'_j - h_j)\}$

Proof:

1) proposal distribution

Let

$$u \sim uniform[-\frac{1}{2}, \frac{1}{2}] \ \ and \ \ log(\frac{h'_j}{h_j}) = u$$

then,

$$\frac{h'_j}{h_j} = e^u$$

$$h'_j = e^u \times h_j$$

$$log(h'_j) = u + log(h_j)$$

Also,

$$P(h'_j \le \alpha) = P(e^u h_j \le \alpha)$$

$$= P(e^u \le \frac{\alpha}{h_j})$$

$$= P(u \le log(\frac{\alpha}{h_j}))$$

$$= \frac{log(\frac{\alpha}{h_j}) - (-\frac{1}{2})}{\frac{1}{2} - (-\frac{1}{2})}$$

$$= log(\frac{\alpha}{h_j}) + \frac{1}{2}$$

then,

$$f(\alpha) = \frac{d}{d\alpha}\left\{log(\frac{\alpha}{h_j}) + \frac{1}{2}\right\}$$

$$= \frac{d}{d\alpha}\left\{log(\frac{1}{h_j}) + log(\alpha) + \frac{1}{2}\right\}$$

$$= \frac{1}{\alpha}$$

therefore,

$$q(h_j, h'_j) = \frac{1}{h'_j}$$

2) acceptance probability

Let $\pi(h_j)$ denote the posterior density,

then,

$$\pi(h_j) = \frac{prior \times likelihood}{\int prior \times likelihood}$$

$$= \frac{p(y_1, ..., y_n|h_j)p(h_j)}{\int p(y_1, ..., y_n|h_j)p(h_j)}$$

then,

$$\frac{\pi(h_j')q(h_j',h_j)}{\pi(h_j)q(h_j,h_j')} = \frac{\frac{p(y_1,...,y_n|h_j')p(h_j')}{\int p(y_1,...,y_n|h_j')p(h_j')} \times \frac{1}{h_j'}}{\frac{p(y_1,...,y_n|h_j)p(h_j)}{\int p(y_1,...,y_n|h_j)p(h_j)} \times \frac{1}{h_j}}$$

$$\propto \frac{p(y_1,...,y_n|h_j')p(h_j') \times h_j}{p(y_1,...,y_n|h_j)p(h_j) \times h_j'}$$

$$= \frac{exp\{(\sum_{i\neq j} m_i \times log(h_i) - \sum_{i\neq j}(s_{i+1}-s_i)\times h_i) + m_j \times log(h_j') - (s_{j+1}-s_j)\times h_j'\} \times \frac{\beta^\alpha}{\Gamma(\alpha)} {h_j'}^{\alpha-1} e^{-\beta h_j'} \times h_j}{exp\{\sum_{i=1}^k m_i \times log(h_i) - \sum_{i=1}^k (s_{i+1}-s_i)h_i\} \times \frac{\beta^\alpha}{\Gamma(\alpha)} {h_j}^{\alpha-1} e^{-\beta h_j} \times h_j'}$$

$$= exp\{(\sum_{i\neq j} m_i \times log(h_i) - \sum_{i\neq j}(s_{i+1}-s_i)\times h_i) + m_j \times log(h_j') - (s_{j+1}-s_j)\times h_j' - \sum_{i=1}^k m_i \times log(h_i) -$$

$$- \sum_{i=1}^k (s_{i+1}-s_i)\times h_i\} \times (\frac{h_j'}{h_j})^{\alpha-2} \times exp\{-\beta(h_j'-h_j)\}$$

$$= exp\{m_j \times (log(h_j') - log(h_j)) - (s_{j+1}-s_j)(h_j'-h_j)\} \times (\frac{h_j'}{h_j})^{\alpha-2} \times exp\{-\beta(h_j'-h_j)\}$$

$$= exp\{m_j \times (log(h_j') - log(h_j)) - (s_{j+1}-s_j)(h_j'-h_j)\} \times exp\{(\alpha-2)(log(h_j') - log(h_j))\} \times exp\{-\beta(h_j'-h_j)\}$$

$$= exp\{m_j \times (log(h_j') - log(h_j)) - (s_{j+1}-s_j)(h_j'-h_j) + (\alpha-2)(log(h_j') - log(h_j)) - \beta(h_j'-h_j)\}$$

$$= exp\{(m_j+\alpha-2)(log(h_j') - log(h_j)) - (s_{j+1}-s_j+\beta)(h_j'-h_j)\}$$

therefore,

$$\alpha(h_j,h_j') = min\{1, \frac{\pi(h_j')q(h_j',h_j)}{\pi(h_j)q(h_j,h_j')}\}$$

$$= min\{1, exp\{(m_j+\alpha-2)(log(h_j') - log(h_j)) - (s_{j+1}-s_j+\beta)(h_j'-h_j)\}\}$$

2. Position

Algorithm:

1) generate j randomly form 2 to k-1, $U \sim unifrom(0,1)$
2) if accepted ($U < \alpha(s_j, s_j')$), then generate $s_j' \sim U(s_{j-1}, s_{j+1})$. Otherwise, set $h_j' = h_j$.

Proposal: $q(s_j, s_j') = \frac{1}{s_{j+1}-s_{j-1}}$

Acceptance probability:

$$\alpha(s_j, s_j') = min\{1, exp\{(m_{j-1}' - m_{j-1}) \times log(h_{j-1}) + (m_j' - m_j) \times log(h_j) + (s_j' - s_j)(h_j - h_{j-1}) + \\ log(s_{j+1} - s_j') + log(s_j' - s_{j-1}) - log(s_{j+1} - s_j) - log(s_j - s_{j-1})\}\}$$

Proof

1) Prior density

$$\begin{aligned}
\pi(s_2, ..., s_k) &= \pi(s_2 \in [t_1, t_3], ..., s_k \in [t_{2k-3}, t_{2k-1}]) \\
&= (2k-1)! \times \pi(s_2 \in [t_1, t_3]) \times ... \times \pi(s_k \in [t_{2k-3}, t_{2k-1}]) \\
&= (2k-1)! \times \prod_{i=2}^{k} \int_{[t_{2i-3}, t_{2i-1}]} f(s_i)^{2k-1} ds_i \\
&= (2k-1)! \times \prod_{i=2}^{k} \int_{[t_{i-1}, t_{i+1}]} \frac{1}{(s_{k+1} - s_1)^{2k-1}} dt_{2i-2} \\
&= (2k-1)! \times \prod_{i=1}^{k} \int_{[s_i, s_{i+1}]} \frac{1}{(s_{k+1} - s_1)^{2k-1}} dt_{2i-1} \\
&= \frac{(2k-1)!}{(s_{k+1} - s_1)^{2k-1}} \times \prod_{i=1}^{k} (s_{i+1} - s_i)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi(s_j | s_2, ...s_{j-1}, s_{j+1}, ..., s_k) &= \frac{1!}{(s_{j+1 - s_{j-1}})} \times (s_{j+1} - s_j)(s_j - s_{j-1}) \\
&= \frac{1}{(s_{j+1 - s_{j-1}})} \times (s_{j+1} - s_j)(s_j - s_{j-1})
\end{aligned}$$

2. acceptance probability

$$\frac{\pi(s_2,...,s'_j,...,s_k)q(s'_j,s_j)}{\pi(s_2,...,s_j,...,s_k)q(s_j,s'_j)} \propto \frac{p(y_1,...,y_n|s'_j)\pi(s'_j|s_2,...s_{j-1},s_{j+1},...,s_k)q(s'_j,s_j)}{p(y_1,...,y_n|s_j)\pi(s_j|s_2,...s_{j-1},s_{j+1},...,s_k)q(s_j,s'_j)}$$

$$= \frac{p(y_1,...,y_n|s'_j)\pi(s'_j|s_2,...s_{j-1},s_{j+1},...,s_k)\frac{1}{s_{j+1}-s_{j-1}}}{p(y_1,...,y_n|s_j)\pi(s_j|s_2,...s_{j-1},s_{j+1},...,s_k)\frac{1}{s_{j+1}-s_{j-1}}}$$

$$= \frac{p(y_1,...,y_n|s'_j)\pi(s'_j|s_2,...s_{j-1},s_{j+1},...,s_k)}{p(y_1,...,y_n|s_j)\pi(s_j|s_2,...s_{j-1},s_{j+1},...,s_k)}$$

$$= \Bigg(\frac{exp\{(\sum_{i\neq j,j-1} m_i \times log(h_i) - \sum_{i\neq j,j-1}(s_{i+1}-s_i)\times h_i) + m'_{j-1}\times log(h_{j-1}) + m'_j \times log(h_j) -}{exp\{\sum_{i=1}^k m_i \times log(h_i) - \sum_{i=1}^k (s_{i+1}-s_i)\times h_i\}}$$

$$\frac{-(s'_j - s_{j-1})\times h_{j-1} - (s_{j+1}-s'_j)\times h_j\}}{exp\{\sum_{i=1}^k m_i \times log(h_i) - \sum_{i=1}^k (s_{i+1}-s_i)\times h_i\}} \Bigg) \times \frac{\frac{1}{(s_{j+1}-s_{j-1})}\times (s_{j+1}-s'_j)(s'_j - s_{j-1})}{\frac{1}{(s_{j+1}-s_{j-1})}\times (s_{j+1}-s_j)(s_j - s_{j-1})}$$

$$= exp\{-m_{j-1}\times log(h_{j-1}) - m_j \times log(h_j) + m'_{j-1}\times log(h_{j-1}) + m'_j \times log(h_j) + (s_{j+1}-s_j)\times h_j +$$

$$+ (s_j - s_{j-1})\times h_{j-1} - (s'_j - s_{j-1})\times h_{j-1} - (s_{j+1}-s'_j)\times h_j\} \times \frac{(s_{j+1}-s'_j)(s'_j - s_{j-1})}{(s_{j+1}-s_j)(s_j - s_{j-1})}$$

$$= exp\{(m'_{j-1} - m_{j-1})\times log(h_{j-1}) + (m'_j - m_j)\times log(h_j) + (s'_j - s_j)\times_j + (s_j - s'_j)\times h_{j-1}\}\times$$

$$\times exp\{log(s_{j+1}-s'_j) + log(s'_j - s_{j-1}) - log(s_{j+1}-s_j) - log(s_j - s_{j-1})\}$$

$$= exp\{(m'_{j-1} - m_{j-1})\times log(h_{j-1}) + (m'_j - m_j)\times log(h_j) + (s'_j - s_j)(h_j - h_{j-1}) +$$

$$+ log(s_{j+1}-s'_j) + log(s'_j - s_{j-1}) - log(s_{j+1}-s_j) - log(s_j - s_{j-1})\}$$

therefore,

$$\alpha(s_j,s'_j) = min\{1, \frac{\pi(s_2,...,s'_j,...,s_k)q(s'_j,s_j)}{\pi(s_2,...,s_j,...,s_k)q(s_j,s'_j)}\}$$

$$= min\{1, exp\{(m'_{j-1} - m_{j-1})\times log(h_{j-1}) + (m'_j - m_j)\times log(h_j) + (s'_j - s_j)(h_j - h_{j-1}) + log(s_{j+1}-s'_j) +$$

$$+ log(s'_j - s_{j-1}) - log(s_{j+1}-s_j) - log(s_j - s_{j-1})\}\}$$

3. Birth step:

parameter: $x = (k, h_1, ...h_k, s_2, ..., s_k)$

Alogrithm

1) randomly choose j from 1 to 2k-1
2) generate $s' \sim uniform(s_j, s_{j+1})$
3) calculate $h'_j = h_j(\frac{u}{1-u})^{\frac{s_{j+1}-s'}{s_{j+1}-s_j}}$ and $h'_{j+1} = (\frac{u}{1-u})^{\frac{s_j-s'}{s_{j+1}-s_j}} \times h_j$
4) if acccepted($U < \alpha(x, x')$),

   update s_{j+1} to s', relabel s_{j+2} as s'_{j+1}, s_{j+3} as $s_{,j+2}$, ... ,

   update h_j to h'_j and h'_{j+1}, relabel h_{j+1} as h'_{j+2}, h_{j+2} as h'_{j+3}, ... ,

   k = k + 1

Prior density: $p(x) = e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} \times \prod_{i=1}^{k} \frac{\beta^\alpha}{\Gamma(\alpha)} {h_i}^{\alpha-1} e^{-\beta h_i} \times \frac{(2k-1)!}{(s_{k+1}-s_1)^{2k-1}} \prod_{i=1}^{k}(s_{i+1} - s_i)$

Acceptance probability:

$$\alpha(x, x') = min\{1, exp\{-m_j log(h_j) + m'_j log(h'_j) + m'_{j+1} log(h'_{j+1}) + (s_{j+1} - s_j)h_j - (s' - s_j)h'_j - (s_{j+1} - s')h'_{j+1} +$$

$$+ \alpha log(\beta) - log(\Gamma(\alpha)) + (\alpha - 1)log(h'_j) + (\alpha - 1)log(h'_{j+1}) - \alpha \times log(h_j) - \beta(h'_j + h'_{j+1} - h'_j) +$$

$$+ log(2\lambda) + log(2k + 1) - log(s_{k+1} - s_1) + log(s' - s_j) + log(s_{j+1} - s') - log(s_{j+1} - s_j) + log(d_k) -$$

$$- log(d_{k-1}) - log(k) + 2 \times log(h'_j + h'_{j+1})\}\}$$

Proof

1) h'_j, h'_{j+1}

Since weighted geometric mean satisfying $(s' - s_j)log(h'_j) + (s_{j+1} - s')log(h'_{j+1}) = (s_{j+1} - s_j)log(h_j)$ and $\frac{h'_{j+1}}{h'_j} = \frac{1-u}{u}$.

then,

$$log({h'}_j^{s'-s_j} \times {h'}_{j+1}^{s_{j+1}-s'}) = log(h_j^{s_{j+1}-s_j})$$

$${h'}_j^{s'-s_j} \times {h'}_{j+1}^{s_{j+1}-s'} = h_j^{s_{j+1}-s_j}$$

$${h'}_j^{s'-s_j} \times (\frac{1-u}{u}h'_j)^{s_{j+1}-s'} = h_j^{s_{j+1}-s_j}$$

$${h'}_j^{s'-s_j+s_{j+1}-s'} \times (\frac{1-u}{u})^{s_{j+1}-s'} = h_j^{s_{j+1}-s_j}$$

$${h'}_j^{s_{j+1}-s_j} \times (\frac{1-u}{u})^{s_{j+1}-s'} = h_j^{s_{j+1}-s_j}$$

$${h'}_j \times (\frac{1-u}{u})^{\frac{s_{j+1}-s'}{s_{j+1}-s_j}} = h_j$$

$${h'}_j = (\frac{u}{1-u})^{\frac{s_{j+1}-s'}{s_{j+1}-s_j}} \times h_j$$

then,

$$\begin{aligned} {h'}_{j+1} &= \frac{1-u}{u}h'_j \\ &= \frac{1-u}{u} \times (\frac{u}{1-u})^{\frac{s_{j+1}-s'}{s_{j+1}-s_j}} \times h_j \\ &= (\frac{u}{1-u})^{-1} \times (\frac{u}{1-u})^{\frac{s_{j+1}-s'}{s_{j+1}-s_j}} \times h_j \\ &= (\frac{u}{1-u})^{\frac{s_{j+1}-s'}{s_{j+1}-s_j}-1} \times h_j \\ &= (\frac{u}{1-u})^{\frac{s_j-s'}{s_{j+1}-s_j}} \times h_j \end{aligned}$$

therefore,

$${h'}_j = (\frac{u}{1-u})^{\frac{s_{j+1}-s'}{s_{j+1}-s_j}} \times h_j {h'}_{j+1} = (\frac{u}{1-u})^{\frac{s_j-s'}{s_{j+1}-s_j}} \times h_j$$

2) acceptance probability

$$\alpha(x, x') = \frac{p(y_1, ..., y_n|x')}{p(y_1, ..., y_n|x)} \times \frac{p(x')}{p(x)} \times \frac{j(x')}{j(x)q_k(u^{(k)})} \times |\frac{\partial(\theta^{(k')})}{\partial(\theta^{(k)}, u^{(k)})}|$$

likelihood ratio

$$\frac{p(y_1,...,y_n|x')}{p(y_1,...,y_n|x)} = \frac{exp\{\sum_{i\neq j} m_i log(h_i) - \sum_{i\neq j}(s_{i+1}-s_i)h_i + m'_j log(h'_j) + m'_{j+1}log(h'_{j+1}) - (s'-s_j)h'_j - (s_{j+1}-s')h'_{j+1}\}}{exp(\{\sum_{i=1}^k m_i log(h_i) - \sum_{i=1}^k (s_{i+1}-s_i)h_i\}}$$

$$= exp\{(\sum_{i\neq j} m_i log(h_i) - \sum_{i=1}^{k} m_i log(h_i)) + m'_j log(h'_j) + m'_{j+1} log(h'_{j+1}) + \sum_{i=1}^{k}(s_{i+1}-s_i)h_i -$$

$$- \sum_{i\neq j}(s_{i+1}-s_i)h_i - (s'-s_j)h'_j - (s_{j+1}-s')h'_{j+1}\}$$

$$= exp\{-m_j log(h_j) + m'_j log(h'_j) + m'_{j+1} log(h'_{j+1}) + (s_{j+1}-s_j)h_j - (s'-s_j)h'_j - (s_{j+1}-s')h'_{j+1}\}$$

prior density ratio

$$\frac{p(x')}{p(x)} = \frac{p(k+1, h_1, ..., h_{j-1}, h'_j, h'_{j+1}, h'_{j+2}(h_{j+1}), ..., h'_{k+1}(h_k), s_2, ..., s_j, s'_{j+1}(s'), s'_{j+2}(s_{j+1}), ..., s'_{k+1}(s_k))}{p(k, h_1, ...h_j, ..., h_k, s_1, ..., s_j, s_{j+1}, ..., s_k)}$$

$$= \frac{e^{-\lambda}\frac{\lambda^k}{k!} \times \prod_{i\neq j} \frac{\beta^\alpha}{\Gamma(\alpha)} {h_i}^{\alpha-1} e^{-\beta h_i} \times \frac{\beta^\alpha}{\Gamma(\alpha)} {h'_j}^{\alpha-1} e^{-\beta h'_j} \times \frac{\beta^\alpha}{\Gamma(\alpha)} {h'_{j+1}}^{\alpha-1} e^{-\beta h'_{j+1}} \times \frac{(2(k+1)-1)!}{(s_{k+1}-s_1)^{2(k+1)-1}} \prod_{i\neq j}(s_{i+1}-s_i)\times}{e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} \times \prod_{i=1}^k \frac{\beta^\alpha}{\Gamma(\alpha)} {h_i}^{\alpha-1} e^{-\beta h_i} \times \frac{(2k-1)!}{(s_{k+1}-s_1)^{2k-1}} \prod_{i=1}^k (s_{i+1}-s_i)}$$

$$\frac{\times(s'-s_j)\times(s_{j+1}-s')}{e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} \times \prod_{i=1}^k \frac{\beta^\alpha}{\Gamma(\alpha)} {h_i}^{\alpha-1} e^{-\beta h_i} \times \frac{(2k-1)!}{(s_{k+1}-s_1)^{2k-1}} \prod_{i=1}^k (s_{i+1}-s_i)}$$

$$= \frac{\lambda}{k} \times \frac{{h'_j}^{\alpha-1} e^{-\beta h'_j} \times \frac{\beta^\alpha}{\Gamma(\alpha)} {h'_{j+1}}^{\alpha-1} e^{-\beta h'_{j+1}}}{h_j^{\alpha-1} e^{-\beta h_j}} \times \frac{(2k+1)(2k)}{(s_{k+1}-s_1)^2} \times \frac{(s'-s_j)(s_{j+1}-s')}{s_{j+1}-s_j}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)}(\frac{h'_j h'_{j+1}}{h_j})^{\alpha-1} e^{-\beta(h'_j + h'_{j+1} - h_j)} \times \frac{(2k+1)(2\lambda)}{(s_{k+1}-s_1)^2} \times \frac{(s'-s_j)(s_{j+1}-s')}{s_{j+1}-s_j}$$

proposal ratio

$$\frac{j(x')}{j(x)q_k(u^{(k)})} = \frac{d_k \times p(s_{j+1})}{b_{k-1} \times p(s') \times \frac{1}{1-0}}$$

$$= \frac{d_k \times \frac{1}{k}}{b_{k-1} \times \frac{1}{s_{k+1}-s_1} \times (\frac{1}{1-0})^k}$$

$$= \frac{d_k \times (s_{k+1}-s_1)}{b_{k-1} \times k}$$

jacobian

Let $v = \frac{s_{j+1}-s'}{s_{j+1}-s_j}$,

then,

$$\left|\frac{\partial(\theta^{(k')})}{\partial(\theta^{(k)}, u^{(k)})}\right| = \left|\frac{\partial(h'_1, ..., h'_{k+1}, s'_2, ..., s'_{k+1})}{\partial(h_1, ..., h_j, u, h_{j+1}, ..., h_k, s_2, ..., s_j, s', s_{j+1}, ..., s_k)}\right|$$

$$= \begin{vmatrix} \frac{\partial H'}{\partial H} & \frac{\partial H'}{\partial S} \\ \frac{\partial S'}{\partial H} & \frac{\partial S'}{\partial S} \end{vmatrix}$$

$$= 1 \times 1 \times \cdots \times \begin{vmatrix} \frac{\partial h'}{\partial h_j} & \frac{\partial h'_j}{\partial u} \\ \frac{\partial h'_{j+1}}{\partial h_j} & \frac{\partial h'_{j+1}}{\partial u} \end{vmatrix} \times 1 \times \cdots \times 1$$

$$= \begin{vmatrix} \frac{\partial h'}{\partial h_j} & \frac{\partial h'_j}{\partial u} \\ \frac{\partial h'_{j+1}}{\partial h_j} & \frac{\partial h'_{j+1}}{\partial u} \end{vmatrix}$$

$$= \begin{vmatrix} (\frac{u}{1-u})^v & h_j \times v(\frac{u}{1-u})^{v-1} \times \frac{1}{(1-u)^2} \\ (\frac{u}{1-u})^{v-1} & h_j \times (v-1)(\frac{u}{1-u})^{v-2} \times \frac{1}{(1-u)^2} \end{vmatrix}$$

$$= |h_j \times (v-1)(\frac{u}{1-u})^{2v-2} \times \frac{1}{(1-u)^2} - h_j \times v(\frac{u}{1-u})^{2v-2} \times \frac{1}{(1-u)^2}|$$

$$= \frac{h_j}{(1-u)^2} \times (\frac{u}{1-u})^{2v-2}$$

$$= \frac{h_j \times u^{2v-2}}{(1-u)^{2v}}$$

$$= \frac{(h_j \times (\frac{u^{v-1}}{(1-u)^v}))^2}{h_j}$$

$$= \frac{(h_j \times (\frac{u^{v-1} \times u + u^{v-1} \times (1-u)}{(1-u)^v}))^2}{h_j}$$

$$= \frac{(h_j \times (\frac{u}{1-u})^v + h_j \times (\frac{u}{1-u})^{v-1})^2}{h_j}$$

$$= \frac{(h'_j + h'_{j+1})^2}{h_j}$$

then,

$$\frac{p(y_1,...,y_n|x')}{p(y_1,...,y_n|x)} \times \frac{p(x')}{p(x)} \times \frac{j(x')}{j(x)q_k(u^{(k)})} \times |\frac{\partial(\theta^{(k')})}{\partial(\theta^{(k)},u^{(k)})}|$$

$$= exp\{-m_j log(h_j) + m'_j log(h'_j) + m'_{j+1} log(h'_{j+1}) + (s_{j+1}-s_j)h_j - (s'-s_j)h'_j - (s_{j+1}-s')h'_{j+1}\}\times$$

$$\times \frac{\beta^\alpha}{\Gamma(\alpha)}(\frac{h'_j h'_{j+1}}{h_j})^{\alpha-1} e^{-\beta(h'_j+h'_{j+1}-h_j)} \times \frac{(2k+1)(2\lambda)}{(s_{k+1}-s_1)^2} \times \frac{(s'-s_j)(s_{j+1}-s')}{s_{j+1}-s_j} \times \frac{d_k \times (s_{k+1}-s_1)}{d_{k-1}\times k}\times$$

$$\times \frac{(h'_j+h'_{j+1})^2}{h_j}$$

$$= exp\{-m_j log(h_j) + m'_j log(h'_j) + m'_{j+1} log(h'_{j+1}) + (s_{j+1}-s_j)h_j - (s'-s_j)h'_j - (s_{j+1}-s')h'_{j+1}+$$

$$+\alpha log(\beta) - log(\Gamma(\alpha)) + (\alpha-1)log(h'_j) + (\alpha-1)log(h'_{j+1}) - (\alpha-1)log(h_j) - \beta(h'_j + h'_{j+1} - h'_j)+$$

$$+log(2\lambda) + log(2k+1) - 2\times log(s_{k+1}-s_1) + log(s'-s_j) + log(s_{j+1}-s') - log(s_{j+1}-s_j) + log(d_k)+$$

$$+log(s_{k+1}-s_1) - log(d_{k-1}) - log(k) + 2\times log(h'_j + h'_{j+1}) - log(h_j)\}$$

$$= exp\{-m_j log(h_j) + m'_j log(h'_j) + m'_{j+1} log(h'_{j+1}) + (s_{j+1}-s_j)h_j - (s'-s_j)h'_j - (s_{j+1}-s')h'_{j+1}+$$

$$+\alpha log(\beta) - log(\Gamma(\alpha)) + (\alpha-1)log(h'_j) + (\alpha-1)log(h'_{j+1}) - \alpha\times log(h_j) - \beta(h'_j + h'_{j+1} - h'_j)+$$

$$+log(2\lambda) + log(2k+1) - log(s_{k+1}-s_1) + log(s'-s_j) + log(s_{j+1}-s') - log(s_{j+1}-s_j) + log(d_k)-$$

$$-log(d_{k-1}) - log(k) + 2\times log(h'_j + h'_{j+1})\}$$

therefore,

$$\alpha(x,x') = min\{1, \frac{p(y_1,...,y_n|x')}{p(y_1,...,y_n|x)} \times \frac{p(x')}{p(x)} \times \frac{j(x')}{j(x)q_k(u^{(k)})} \times |\frac{\partial(\theta^{(k')})}{\partial(\theta^{(k)},u^{(k)})}|\}$$

$$= min\{1, exp\{-m_j log(h_j) + m'_j log(h'_j) + m'_{j+1} log(h'_{j+1}) + (s_{j+1}-s_j)h_j - (s'-s_j)h'_j - (s_{j+1}-s')h'_{j+1}+$$

$$+\alpha log(\beta) - log(\Gamma(\alpha)) + (\alpha-1)log(h'_j) + (\alpha-1)log(h'_{j+1}) - \alpha\times log(h_j) - \beta(h'_j + h'_{j+1} - h'_j)+$$

$$+log(2\lambda) + log(2k+1) - log(s_{k+1}-s_1) + log(s'-s_j) + log(s_{j+1}-s') - log(s_{j+1}-s_j) + log(d_k)-$$

$$-log(d_{k-1}) - log(k) + 2\times log(h'_j + h'_{j+1})\}\}$$

4. Death step

parameter: $x = (k, h_1, ...h_k, s_2, ..., s_k)$

Alogrithm:

1) randomly choose j from 1 to 2k-1
2) calculate $h'_j = h_j^{\frac{s_{j+1}-s_j}{s_{j+2}-s_j}} \times h_{j+1}^{\frac{s_{j+2}-s_{j+1}}{s_{j+2}-s_j}}$

4) if accepted ($U < \alpha(x, x')$),

   remove s_{j+1}, relabel s_{j+2} as s'_{j+1}, s_{j+3} as $s_{,j+2}, \ldots$,

   update h_j and h_{j+1} to h'_j, relabel h_{j+2} as h'_{j+1}, h_{j+3} as $h'_{j+2}, \ldots$,

   k = k - 1

Acceptance probability:

$$\alpha(x, x') = min\{1, \frac{p(y_1, ..., y_n|x')}{p(y_1, ..., y_n|x)} \times \frac{p(x')}{p(x)} \times \frac{j(x')}{j(x)q_k(u^{(k)})} \times |\frac{\partial(\theta^{(k')})}{\partial(\theta^{(k)}, u^{(k)})}|\}$$

$$= min\{1, exp\{-m_{j+1}log(h_{j+1}) - m_j log(h_j) + m'_j log(h'_j) + (s_{j+2} - s_{j+1})h_{j+1} + (s_{j+1} - s_j)h_j - (s_{j+2} - s_j)h'_j +$$

$$+ log(\Gamma(\alpha)) - \alpha \times log(\beta) + (\alpha - 1)log(h'_j) - (\alpha - 1)log(h_j) - (\alpha - 1)log(h_{j+1}) + \beta(h_j + h_{j+1} - h'_j) +$$

$$+ log(s_{j+2} - s_j) - log(s_{j+1} - s_j) - log(s_{j+2} - s_{j+1}) + log(b_{k-1}) + log(k) - log(d_k) - log(s_{k+1} - s_1) +$$

$$+ log(h'_j) - 2 \times log(h_j + h_{j+1})\}\}$$

Proof

1) h'_j

Since weighted geometric mean satisfying $(s_{j+1} - s_j)log(h_j) + (s_{j+2} - s_j)log(h_{j+1}) = (s'_{j+1} - s'_j)log(h'_j)$,

then,

$$log(h_j^{s_{j+1}-s_j}) + log(h_{j+1}^{s_{j+2}-s_j}) = log({h'}_j^{s'_{j+1}-s'_j})$$

$$log(h_j^{s_{j+1}-s_j} \times h_{j+1}^{s_{j+2}-s_j}) = log({h'}_j^{s'_{j+1}-s'_j})$$

$$h_j^{s_{j+1}-s_j} \times h_{j+1}^{s_{j+2}-s_j} = {h'}_j^{s'_{j+1}-s'_j}$$

$$h'_j = h_j^{\frac{s_{j+1}-s_j}{s'_{j+1}-s'_j}} \times h_{j+1}^{\frac{s_{j+2}-s_j}{s'_{j+1}-s'_j}}$$

$$h'_j = h_j^{\frac{s_{j+1}-s_j}{s_{j+2}-s_j}} \times h_{j+1}^{\frac{s_{j+2}-s_j}{s_{j+2}-s_j}}$$

2) acceptance probability

likelihood ratio

$$\frac{p(y_1,...,y_n|x')}{p(y_1,...,y_n|x)} = \frac{exp\{\sum_{i\neq j,j+1} m_i \times log(h_i) - \sum_{i\neq j,j+1}(s_{i+1}-s_i)\times h_i + m'_j \times log(h'_j) - (s_{j+2-s_j}) \times h'_j\}}{exp\{\sum_{i=1}^{k} m_i \times log(h_i) - \sum_{i=1}^{k}(s_{i+1}-s_i)\times h_i\}}$$

$$= exp\{\sum_{i\neq j,j+1} m_i \times log(h_i) - \sum_{i\neq j,j+1}(s_{i+1}-s_i)\times h_i + m'_j \times log(h'_j) - (s_{j+2-s_j}) \times h'_j - \sum_{i=1}^{k} m_i \times log(h_i)$$

$$+ \sum_{i=1}^{k}(s_{i+1}-s_i)\times h_i\}$$

$$= exp\{-m_{j+1} \times log(h_{j+1}) - m_j \times log(h_j) + m'_j \times log(h'_j) + (s_{j+2}-s_{j+1}) \times h_{j+1} + (s_{j+1} - s_j \times)h_j$$

$$-(s_{j+2}-s_j) \times h'_j\}$$

prior density ratio

$$\frac{p(x')}{p(x)} = \frac{p(k-1, h_1, ..., h_{j-1}, h'_j, h'_{j+1}(h_{j+2}), h'_{j+2}(h_{j+3}), ..., h'_{k-1}(h_k), s_2, ..., s_j, s'_{j+1}(s_{j+2}), s'_{j+2}(s_{j+3}), ..., s'_{k-1}(s_k))}{p(k, h_1, ...h_j, ..., h_k, s_1, ..., s_j, s_{j+1}, ..., s_k)}$$

$$= \frac{e^{-\lambda}\frac{\lambda^{k-2}}{(k-2)!} \times \prod_{i\neq j,j+1} \frac{\beta^\alpha}{\Gamma(\alpha)} {h_i}^{\alpha-1} e^{-\beta h_i} \times \frac{\beta^\alpha}{\Gamma(\alpha)} {h'_j}^{\alpha-1} e^{-\beta h'_j} \times \frac{(2(k-1)-1)!}{(s_{k+1}-s_1)^{2(k-1)-1}} \prod_{i\neq j,j+1}(s_{i+1}-s_i) \times (s_{j+2}-s_j)}{e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} \times \prod_{i=1}^k \frac{\beta^\alpha}{\Gamma(\alpha)} {h_i}^{\alpha-1} e^{-\beta h_i} \times \frac{(2k-1)!}{(s_{k+1}-s_1)^{2k-1}} \prod_{i=1}^k (s_{i+1}-s_i)}$$

$$= \frac{k-1}{\lambda} \times \frac{{h'_j}^{\alpha-1} e^{-\beta h'_j}}{h_j^{\alpha-1} e^{-\beta h_j} \times \frac{\beta^\alpha}{\Gamma(\alpha)} {h_{j+1}}^{\alpha-1} e^{-\beta h_{j+1}}} \times \frac{(s_{k+1}-s_1)^2}{(2k-1)(2k-2)} \times \frac{s_{j+2}-s_j}{(s_{j+1}-s_j)(s_{j+2}-s_{j+1})}$$

$$= \frac{\Gamma(\alpha)}{\beta^\alpha} (\frac{h'_j}{h_j \times h_{j+1}})^{\alpha-1} e^{-\beta(h'_j - h_j - h_{j+1})} \times \frac{(s_{k+1}-s_1)^2}{(2k-1)(2\lambda)} \times \frac{s_{j+2}-s_j}{(s_{j+1}-s_j)(s_{j+2}-s_{j+1})}$$

proposal ratio

$$\frac{j(x')q(u_k^{(k')})}{j(x)} = \frac{b_{k-2} \times p(s') \times \frac{1}{1-0}}{d_{k-1} \times p(s_{j+1})}$$

$$= \frac{b_{k-2} \times \frac{1}{s_{k+1}-s_1}}{d_{k-1} \times \frac{1}{k-1}}$$

$$= \frac{b_{k-2} \times (k-1)}{d_{k-1} \times (s_{k+1}-s_1)}$$

Jacobian

Let $v = \frac{s_{j+1}-s_j}{s_{j+2}-s_j}$,

then,

$h'_j = h_j^v \times h_{j+1}^{1-v}$

then,

$$\left|\frac{\partial(\theta^{(k')}, u^{(k')})}{\partial(\theta^{(k)}}\right| = \left|\frac{\partial(h'_1, ..., h'_{j-1}, h'_j, u, h'_{j+1}, ..., h'_{k-1}, s'_2, ..., s'_j, s', s'_{j+1}, ..., s'_{k-1})}{\partial(h_1, ..., h_{j-1}, h_j, h_{j+1}, h_{j+2}, ..., h_k, s_2, ..., s_j, s_{j+1}, s_{j+2}, ..., s_k)}\right|$$

$$= \begin{vmatrix} \frac{\partial H'}{\partial H} & \frac{\partial H'}{\partial S} \\ \frac{\partial S'}{\partial H} & \frac{\partial S'}{\partial S} \end{vmatrix}$$

$$= 1 \times 1 \times \cdots \times \begin{vmatrix} \frac{\partial h'_j}{\partial h_j} & \frac{\partial h'_j}{\partial h_{j+1}} \\ \frac{\partial u}{\partial h_j} & \frac{\partial u}{\partial h_{j+1}} \end{vmatrix} \times 1 \times \cdots \times 1$$

$$= \begin{vmatrix} \frac{\partial h'_j}{\partial h_j} & \frac{\partial h'_j}{\partial h_{j+1}} \\ \frac{\partial u}{\partial h_j} & \frac{\partial u}{\partial h_{j+1}} \end{vmatrix}$$

$$= \begin{vmatrix} (\frac{1-u}{u})^{1-v} & (\frac{u}{1-u})^{v} \\ (\frac{h_{j+1}}{(h_{j+1}+h_j)^2} & \frac{-h_j}{(h_{j+1}+h_j)^2} \end{vmatrix}$$

$$= \left|(\frac{1-u}{u})^{1-v} \times \frac{-h_j}{(h_{j+1}+h_j)^2} - \frac{h_{j+1}}{(h_{j+1}+h_j)^2} \times (\frac{u}{1-u})^{v}\right|$$

$$= \left|(\frac{h_{j+1}}{h_j})^{1-v} \times \frac{-h_j}{(h_{j+1}+h_j)^2} - \frac{h_{j+1}}{(h_{j+1}+h_j)^2} \times (\frac{h_j}{h_{j+1}})^{v}\right|$$

$$= \left|\frac{1}{(h_{j+1}+h_j)^2}\left(-h_{j+1}^{1-v} \times h_j^v - h_{j+1}^{1-v} \times h_j^v\right)\right|$$

$$= \left|\frac{-2h'_j}{(h_{j+1}+h_j)^2}\right|$$

$$= \frac{2h'_j}{(h_{j+1}+h_j)^2}$$

then,

$$\frac{p(y_1,...,y_n|x')}{p(y_1,...,y_n|x)} \times \frac{p(x')}{p(x)} \times \frac{j(x')}{j(x)q_k(u^{(k)})} \times |\frac{\partial(\theta^{(k')})}{\partial(\theta^{(k)},u^{(k)})}|$$

$$= exp\{-m_{j+1} \times log(h_{j+1}) - m_j \times log(h_j) + m'_j \times log(h'_j) + (s_{j+2}-s_{j+1}) \times h_{j+1} + (s_{j+1}-s_j) \times h_j -$$

$$-(s_{j+2}-s_j) \times h'_j\} \times \frac{\Gamma(\alpha)}{\beta^\alpha}(\frac{h'_j}{h_j \times h_{j+1}})^{\alpha-1}e^{-\beta(h'_j-h_j-h_{j+1})} \times \frac{(s_{k+1}-s_1)^2}{(2k-1)(2\lambda)} \times \frac{s_{j+2}-s_j}{(s_{j+1}-s_j)(s_{j+2}-s_{j+1})}\times$$

$$\times \frac{b_{k-2} \times (k-1)}{d_{k-1} \times (s_{k+1}-s_1)} \times \frac{2h'_j}{(h_{j+1}+h_j)^2}$$

$$= exp\{-m_{j+1} \times log(h_{j+1}) - m_j \times log(h_j) + m'_j \times log(h'_j) + (s_{j+2}-s_{j+1}) \times h_{j+1} + (s_{j+1}-s_j) \times h_j -$$

$$-(s_{j+2}-s_j) \times h'_j + log(\Gamma(\alpha)) - \alpha \times log(\beta) + (\alpha-1)log(h'_j) - (\alpha-1)log(h_j) - (\alpha-1)log(h_{j+1})$$

$$-\beta(h'_j-h_j-h_{j+1}) + log(s_{k+1}-s_1) - log(2k-1) - log(2\lambda) + log(s_{j+2}-s_j) - log(s_{j+1}-s_j) - log(s_{j+2}-s_{j+1})$$

$$+ log(b_{k-2}) + log(k-1) - log(d_{k-1}) + 2 \times log(h'_j) - 2 \times log(h_j+h_{j+1})\}$$

$$= exp\{-m_{j+1} \times log(h_{j+1}) - m_j \times log(h_j) + m'_j \times log(h'_j) + (s_{j+2}-s_{j+1}) \times h_{j+1} + (s_{j+1}-s_j) \times h_j -$$

$$-(s_{j+2}-s_j) \times h'_j + log(\Gamma(\alpha)) - \alpha \times log(\beta) + \alpha \times log(h'_j) - (\alpha-1)log(h_j) - (\alpha-1)log(h_{j+1})$$

$$-\beta(h'_j-h_j-h_{j+1}) + log(s_{k+1}-s_1) - log(2k-1) - log(\lambda) + log(s_{j+2}-s_j) - log(s_{j+1}-s_j) - log(s_{j+2}-s_{j+1})$$

$$+ log(b_{k-2}) + log(k-1) - log(d_{k-1}) - 2 \times log(h_j+h_{j+1})\}$$

therefore,

$$\alpha(x,x') = min\{1, \frac{p(y_1,...,y_n|x')}{p(y_1,...,y_n|x)} \times \frac{p(x')}{p(x)} \times \frac{j(x')}{j(x)q_k(u^{(k)})} \times |\frac{\partial(\theta^{(k')})}{\partial(\theta^{(k)},u^{(k)})}|\}$$

$$= min\{1, exp\{-m_{j+1}log(h_{j+1}) - m_jlog(h_j) + m'_jlog(h'_j) + (s_{j+2}-s_{j+1})h_{j+1} + (s_{j+1}-s_j)h_j - (s_{j+2}-s_j)h'_j +$$

$$+ log(\Gamma(\alpha)) - \alpha \times log(\beta) + (\alpha-1)log(h'_j) - (\alpha-1)log(h_j) - (\alpha-1)log(h_{j+1}) + \beta(h_j+h_{j+1}-h'_j) +$$

$$+ log(s_{j+2}-s_j) - log(s_{j+1}-s_j) - log(s_{j+2}-s_{j+1}) + log(b_{k-1}) + log(k) - log(d_k) - log(s_{k+1}-s_1) +$$

$$+ log(h'_j) - 2 \times log(h_j+h_{j+1})\}\}$$