Stochastic Justification of Some Simple Reliability Models*

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Abstract

In this paper we derive conditions on the internal wear process under which the resulting time-to-failure model will be of the simple collapsible form (Oakes, 1995, Duchesne and Lawless, 2000) when the usage accumulation history is available.

We suppose that failure occurs when internal wear crosses a certain threshold and/or a traumatic event causes the item to fail (Cox, 1999 and Bagdonavičius and Nikulin, 2001). We model the infinitesimal increment in internal wear as a function of time, accumulated internal wear and usage history, and we derive conditions on this function to get a collapsible model for the distribution of time-to-failure given the usage history.

We reach the conclusion that collapsible models form the subset of accelerated failure time models with time-varying covariates (Robins and Tsiatis, 1992) for which the time transformation function satisfies certain simple properties.

Keywords: accelerated failure time model, additive hazard model, collapsible model, degradation, differential equation, diffusion, gamma process, internal wear, traumatic event, usage rate.

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1 Introduction

In the recent literature, several methods and models have been suggested to include the effect of the usage history on the lifetime distribution of various items. Even though some multivariate models for the time and usage to failure have been proposed (Kordonsky and Gertsbakh,

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1997, Singpurwalla and Wilson, 1998), there seems to be more focus on conditional models for the time to failure distribution given a usage history (Oakes, 1995, Bagdonavičius and Nikulin, 1997, Finkelstein, 1999, Duchesne and Lawless, 2000). One such conditional model is the accelerated failure time (AFT) model with time-varying covariates (Cox and Oakes, 1984, Nelson, 1990, Robins and Tsiatis, 1992, Bagdonavičius and Nikulin, 1997, Finkelstein, 1999). The AFT model is quite popular in reliability, both because of its interpretability, its mathematical properties and its consistency with some engineering/physical principles. Recently, the collapsible model (Oakes, 1995, Kordonsky and Gertsbakh, 1997, Duchesne and Lawless, 2000) was introduced. An attractive feature of the collapsible model is that it allows for a very easy interpretation of the effect of usage on lifetime. Furthermore, it is convenient on a mathematical standpoint as it permits semiparametric or completely non-parametric modelling (Duchesne, 2000). The assumptions made by the model are, however, much stronger than that of the AFT model and a formal omnibus test of the "collapsibility assumption" remains an open problem. For these reasons, it is of interest to investigate the types of failure mechanisms that are consistent with collapsible models, thereby justifying the use of these models in environments where such failure mechanisms are plausible.

In this paper, we describe some stochastic failure mechanisms that can give rise to collapsible models for the conditional distribution of the time to failure given a usage history. This type of approach has been used by several authors in reliability and is becoming popular in biostatistics as well (see Aalen and Gjessing, 2001, and references therein). Our work will follow more along the lines of Singpurwalla (1995) and Cox (1999), who consider various strategies to model the failure mechanism via the item's so-called *internal wear* (or *degradation*) process: (i) failure occurs when the internal wear of the item reaches a certain threshold; (ii) internal wear defines the failure rate of the item, as is the case when a traumatic event (shock) "kills" the item, with the hazard of such a traumatic event increasing with internal wear. We also draw some inspiration from the work of Bagdonavičius and Nikulin (2001) on the modelling of degradation processes when covariates such as usage history are present, though we will model the effect of usage and time on the value of the increment of the wear process, not through a change in the time index of the process.

The paper is organized as follows. Throughout the paper, we consider models where the infinitesimal increment in internal wear is modeled as a function of (subsets of) time, usage accumulation history, and cumulative wear. In Section 2, we introduce some notation and give a precise formulation of the conditional models of interest. We look at how these models may arise when internal wear is a deterministic function of time for a given usage history in Section 3. Similar results are then derived in the case of a stochastic wear process in Section 4. In Section 5, we introduce failures due to traumatic events. We give concluding remarks and ideas for further research in Section 6.

2 Definitions and notation

Let $\{X(t),\ t \geq 0\}$ be the internal wear process of the item. We assume throughout that $X(0) = 0$ and that $\{X(t)\}$ has right-continuous paths with finite left-hand limits w.p. 1. Let X^* be the threshold random variable, with $P[X^* > 0] = 1$. We define the time to failure of an item as $T = \inf\{t :\ X(t) \geq X^*\}$. Finally, let $\{\theta(t),\ t \geq 0\}$ be the usage rate of the item at time t and $y(t) = \int_0^t \theta(u)\ du$ be the cumulative usage at time t. We assume that $\theta(t) \geq 0$ for all $t \geq 0$. For convenience, we let $\boldsymbol{\theta}_t \equiv \{\theta(s), 0 \leq s \leq t\}$ and $\boldsymbol{\theta} \equiv \{\theta(t), t \geq 0\}$. We let $\boldsymbol{\Theta}_t$ and $\boldsymbol{\Theta}$ represent the spaces of all possible usage histories $\boldsymbol{\theta}_t$ and $\boldsymbol{\theta}$, respectively, with $\theta(t)$ piecewise smooth, i.e. for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ there exists a countable set of time points $0 \leq t_1 < t_2 < \ldots$ with $t_i \to \infty$, such that $\theta(t) = a_i(t), t_i < t < t_{i+1}$ where a_i is continuous and smooth over $[t_i, t_{i+1}]$, $i = 1, 2, \ldots$. This family of usage histories is broad enough to include stepwise continuous usage histories (such as on-off or low intensity-high intensity usage patterns), but it excludes cases where $y(t)$ is the realization of a counting process, such as the cumulative number of startups.

Note that intrinsic to any model conditioning on the usage history is the assumption that as a process, usage evolves independently of the internal wear process, i.e.

$$P[\theta(t + \Delta t) \in A | \theta(s), X(s),\ 0 \leq s \leq t] = P[\theta(t + \Delta t) \in A | \boldsymbol{\theta}_t]$$

for any measurable set A. We also assume that X^* is independent of both processes $\{\theta(t)\}$ and $\{X(t)\}$.

We now define the conditional models of interest. Our focus is on collapsible models. Nevertheless, we first define the AFT model that will be used for comparison purposes in our development. We give the definition of Robins and Tsiatis (1992):

Definition 2.1 *The accelerated failure time model is given by*

$$P[T > t|\theta_t] = G\left[\int_0^t \psi(\boldsymbol{\theta}_u; \beta)\, du\right], \tag{1}$$

where $\psi(\cdot, \cdot; \beta)$ is a positive $\boldsymbol{\Theta} \rightarrow [0, \infty)$ map, called time transformation function, that may depend on a vector of unknown parameters β, and $G[\cdot]$ is a survivor function.

Obviously, (1) is too general to be useful in practice. One popular specification of (1) is the log-linear model

$$P[T > t|\boldsymbol{\theta}_t] = G\left[\int_0^t \exp\{\beta g_\theta[\theta(u)]\}\ du\right], \tag{2}$$

where $g_\theta[\cdot]$ is a completely specified 1-1 $[0, \infty) \rightarrow [0, \infty)$ map.

The class of models we are mainly interested in is defined as follows:

Definition 2.2 *A collapsible model is a model described by*

$$P[T > t|\boldsymbol{\theta}_t] = G[\phi(t, y(t); \beta)], \tag{3}$$

where $y(t)$ is the cumulative usage, $G[\cdot]$ is a survivor function, and $\phi(\cdot, \cdot; \beta)$ is a positive $[0, \infty)^2 \rightarrow [0, \infty)$ map, possibly depending on a vector of unknown parameters β, such that $\phi(t, y(t); \beta)$ is non-decreasing in t for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

The function ϕ can be viewed as a common scale in which the age of all the items can be compared, regardless of their usage history. This type of time scale is sometimes referred to as *ideal time scale* (Duchesne and Lawless, 2000), *virtual age* (Finkelstein, 1999), *load-invariant scale* (Kordonsky and Gertsbakh, 1997) or *intrinsic scale* (Cinlar and Ozekici, 1987).

The meaning of the collapsible model (3) is that survival past a certain time point t only depends on t itself and the cumulative usage at that time, not on the entire usage history up to t, i.e. the conditional survival probability only depends on the usage history $\boldsymbol{\theta}_t$ through t and $y(t)$. Some forms of the ideal time scale ϕ in collapsible models lead to nice interpretations. For instance, if $\phi(t, y(t); \beta) = \beta_0 t + \beta_1 y(t)$, then living one time unit has the same effect on the item as β_0/β_1 units of usage (Oakes, 1995, Kordonsky and Gertsbakh, 1997). A similar interpretation holds on a log scale when $\phi(t, y(t)) = t^{\beta_0} y(t)^{\beta_1}$ (Duchesne and Lawless, 2000). One physical/physiological condition is obvious just by looking at the formulation of a simple collapsible model: the damage inflicted by time and usage to an item has to be cumulative and permanent for the model to hold. Note that this must also be true in the case of the log-linear AFT model.

Before considering specific models, we give a very useful result.

Theorem 2.1 *If a model is collapsible for any fixed threshold $X^* = x^* > 0$, and $X(t)$ is non-decreasing as a function of t, then it is also collapsible under a positive random threshold X^*.*

Proof: Let the cumulative distribution function of X^* be F_{X^*}. From the definition of a collapsible model, we want the conditional survival probability $P[T > t|\boldsymbol{\theta}_t]$ to depend on $\boldsymbol{\theta}_t$ only though t and $y(t)$. Because $X(t)$ is non-decreasing, and X^* is independent of $\{X(t)\}$ and $\{\theta(t)\}$, we have that

$$P[T > t|\boldsymbol{\theta}_t] = P[X(t) < X^*|\boldsymbol{\theta}_t] = \int_0^\infty P[X(t) < x^*|\boldsymbol{\theta}_t] dF_{X^*}(x^*).$$

But if the model is collapsible for any fixed threshold x^*, we have that $P[X(t) < x^*|\boldsymbol{\theta}_t] = f(t, y(t), x^*)$ for some function f. Hence, the conditional survival probability only depends on $\boldsymbol{\theta}_t$ through t and $y(t)$. □

3 Deterministic environment

We first start by considering an environment where, conditional on a usage history $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, the internal wear is a deterministic function of time. This may not be quite realistic, but it makes the developments of the subsequent sections more transparent. To emphasize the deterministic nature of the model, we here write $X(t)$ as $x(t)$.

Theorem 2.1 allows us to assume without loss of generality that the threshold $X^* = x^*$, a positive constant, w.p. 1. Let us assume that the wear process, given a usage history, is deterministic and can be described by a differential equation of the form

$$dx(t) = \mu[t, \boldsymbol{\theta}_t, x(t)]dt \tag{4}$$

with initial condition $x(0) = 0$. We assume throughout this paper that μ is a non-negative-valued function. The question of interest is what type of function μ corresponds to the collapsible model?

Let us first write down the conditional survivor function of T. Since μ is non-negative, $x(t)$ will be non-decreasing in t and thus $T > t$ if, and only if, $x(t) < x^*$. Hence,

$$P[T > t|\boldsymbol{\theta}] = P[x(t) < x^*] = I[x^* > x(t)]$$

where $I[\cdot]$ is the indicator function.

We first consider the case where μ in (4) only depends on t and $\boldsymbol{\theta}_t$, i.e. the increment in wear caused by usage and time only depends on time and the usage accumulation history up to that time, not on the accumulated wear; as we shall see from Corollary 3.3, this is a rather weak assumption. Mathematically, we assume that

$$\mu[t, \boldsymbol{\theta}_t, x(t)] = \mu[t, \boldsymbol{\theta}_t]. \tag{5}$$

Note that this defines the AFT model, with $\psi \equiv \mu$ in (1). We want conditions on $\mu[\cdot, \cdot]$ to obtain a collapsible model.

From Definition 2.2, we have that for any fixed threshold x^*, $\mu[\cdot, \cdot]$ must be such that there exists a function f^* such that $I[x^* > x(t)] = f^*[t, y(t)]$ for all $t \geq 0$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Because this must be true for any fixed $x^* > 0$, then $\mu[\cdot, \cdot]$ must be such that there exists a function f with $x(t) = f[t, y(t)]$ for each $t \geq 0$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Before we are able to obtain our main result we need the following three lemmas. The first lemma follows from the Fundamental Theorem of Calculus, since $y(t) = \int_0^t \theta(s)\, ds$.

Lemma 3.1 *We have $y'(s) = \theta(s)$ wherever θ is continuous.*

Lemma 3.2 *Let $\mu_1, \mu_2 : [0,\infty)^2 \to [0,\infty)$ be continuously differentiable functions such that $\frac{\partial}{\partial y}\mu_1(x,y) = \frac{\partial}{\partial x}\mu_2(x,y)$ for all $x, y \geq 0$. Suppose for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, we have $\mu[t, \boldsymbol{\theta}_t] = \mu_1[t, y(t)] + \mu_2[t, y(t)]\,\theta(t)$ wherever $\theta(t)$ is continuous. Let $x(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s] ds$. Then there is a function $f : [0,\infty)^2 \to [0,\infty)$ such that $x(t) = f(t, y(t))$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and all times $t \geq 0$.*

Proof: Define $f : [0,\infty)^2 \to [0,\infty)$ by

$$f(x,y) = \int_0^x \mu_1(s,y)\,ds + \int_0^y \mu_2(0,u)\,du\,.$$

Then by the Fundamental Theorem of Calculus, $\frac{\partial f}{\partial x}(x,y) = \mu_1(x,y)$. Also, since $\frac{\partial \mu_1}{\partial y}$ is continuous, it is bounded on $[0,x]$, so we have $\frac{\partial}{\partial y}\int_0^x \mu_1(s,y)\,ds = \int_0^x \frac{\partial \mu_1}{\partial y}(s,y)\,ds$ (see e.g. Rosenthal, 2000, Proposition 9.2.1). Hence,

$$\frac{\partial f}{\partial y}(x,y) \;=\; \int_0^x \frac{\partial \mu_1}{\partial y}(s,y)\,ds + \mu_2(0,y)$$

$$=\; \int_0^x \frac{\partial \mu_2}{\partial s}(s,y)\,ds + \mu_2(0,y) \;=\; \mu_2(x,y)\,.$$

We then compute (using the chain rule and the Fundamental Theorem of Calculus again, together with Lemma 3.1) that

$$x(t) \;=\; \int_0^t \mu(s, \boldsymbol{\theta}_s)\,ds$$

$$=\; \int_0^t \Big(\mu_1(s,y(s)) + \mu_2(s,y(s))\theta(s)\Big)\,ds$$

$$=\; \int_0^t \Big(\frac{\partial f}{\partial x}(s,y(s)) + \frac{\partial f}{\partial y}(s,y(s))\,y'(s)\Big)\,ds$$

$$=\; \int_0^t \frac{\partial}{\partial s} f(s,y(s))\,ds$$

$$=\; f(t,y(t)) - f(0,0) \;=\; f(t,y(t))\,,$$

as claimed. □

Definition 3.1 *A function $\mu : [0,\infty) \times \boldsymbol{\Theta} \to [0,\infty)$ is regular if the mapping $s \mapsto \mu[s, \boldsymbol{\theta}_s]$ is continuous [resp. continuously differentiable] at $s = t$ whenever the mapping $s \to \boldsymbol{\theta}_s$ is continuous [resp. continuously differentiable] at $s = t$.*

To avoid technical difficulties, we shall always assume that μ is regular.

Lemma 3.3 *Suppose $x(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s]\, ds$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and all $t \geq 0$, where the function $\mu : [0,\infty) \times \boldsymbol{\Theta} \to [0,\infty)$ is regular. Suppose further that there is a function $f : [0,\infty)^2 \to [0,\infty)$ such that $x(t) = f(t, y(t))$ for all $t \geq 0$ and all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Then there are continuously differentiable functions $\mu_1, \mu_2 : [0,\infty)^2 \to [0,\infty)$ such that $\frac{\partial}{\partial y}\mu_1(x,y) = \frac{\partial}{\partial x}\mu_2(x,y)$ for all $x, y \geq 0$, such that*

$$\mu[t, \boldsymbol{\theta}_t] = \mu_1[t, y(t)] + \mu_2[t, y(t)]\, \theta(t)$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and for all $t \geq 0$ such that the usage function θ is continuously differentiable at t.

Proof: Let $g_{\boldsymbol{\theta}}(s) = \mu[s, \boldsymbol{\theta}_s]$. Then $f(t, y(t)) = \int_0^t g_{\boldsymbol{\theta}}(s)\, ds$.

Restrict first to usage functions $\theta : [0,\infty) \to [0,\infty)$ which are continuously differentiable everywhere. Then $g_{\boldsymbol{\theta}}$ is also continuously differentiable, and by the Fundamental Theorem of Calculus, we have $\frac{d}{dt} f(t, y(t)) = g_{\boldsymbol{\theta}}(t)$ for all t. In particular, the mapping $t \to f(t, y(t))$ is twice continuously differentiable. Since this holds for all continuously differentiable usage functions $\theta : [0,\infty) \to [0,\infty)$, it follows (similar to Theorem 4 of Chapter 6 of Marsden, 1974) that f itself is twice continuously differentiable as a function from $[0,\infty)^2$ to $[0,\infty)$. Let $f^{(1)}(x,y) = \frac{\partial}{\partial x} f(x,y)$ and $f^{(2)}(x,y) = \frac{\partial}{\partial y} f(x,y)$. It then follows (as in Theorem 9 of Chapter 6 of Marsden, 1974) that the mixed partials of f must be equal, i.e. $\frac{\partial}{\partial y} f^{(1)}(x,y) = \frac{\partial}{\partial x} f^{(2)}(x,y)$.

We conclude that the function $f : [0,\infty)^2 \to [0,\infty)$ is twice continuously differentiable, with $\frac{\partial}{\partial y} f^{(1)}(x,y) = \frac{\partial}{\partial x} f^{(2)}(x,y)$.

We now consider general usage functions $\theta : [0,\infty) \to [0,\infty)$ (not necessarily continuously differentiable). Suppose the function $\theta : [0,\infty) \to [0,\infty)$ is continuous at t for some particular time t. Then by the chain rule, the function $g_{\boldsymbol{\theta}} : [0,\infty) \to [0,\infty)$ is also continuous at t. Hence, by the Fundamental Theorem of Calculus, since $f(t, y(t)) = \int_0^t g_{\boldsymbol{\theta}}(s)\, ds$, we again have $\frac{d}{dt} f(t, y(t)) = g_{\boldsymbol{\theta}}(t)$ for this particular t. Hence, by the chain rule and Lemma 3.1,

$$g_{\boldsymbol{\theta}}(t) = \frac{d}{dt} f(t, y(t)) = f^{(1)}(t, y(t)) + f^{(2)}(t, y(t)) y'(t)$$

$$= f^{(1)}(t, y(t)) + f^{(2)}(t, y(t))\theta(t)\,.$$

Setting $\mu_1(x, y) = f^{(1)}(x, y)$ and $\mu_2(x, y) = f^{(2)}(x, y)$, the stated conclusion now follows. □

Combining the above two lemmas, we conclude the following.

Theorem 3.1 *Suppose $x(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s]\, ds$, where the function $\mu : [0, \infty) \times \boldsymbol{\Theta} \to [0, \infty)$ is regular. Then there exists a function $f : [0, \infty)^2 \to [0, \infty)$ with $x(t) = f(t, y(t))$ for all $t \geq 0$ and all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, if and only if there exist continuously differentiable functions $\mu_1, \mu_2 : [0, \infty)^2 \to [0, \infty)$ with $\frac{\partial}{\partial y}\mu_1(x, y) = \frac{\partial}{\partial x}\mu_2(x, y)$ for all $x, y \in [0, \infty)$, such that for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and almost every (Lebesgue) $t \geq 0$, we have*

$$\mu[t, \boldsymbol{\theta}_t] = \mu_1[t, y(t)] + \mu_2[t, y(t)]\,\theta(t)\,.$$

A special case of interest is the one where the increment in wear depends only on time and the usage rate at that time.

Corollary 3.1 *If wear is accumulated according to equation (5) with $\mu[t, \boldsymbol{\theta}_t; \beta] \equiv \mu[t, \theta(t); \beta]$ for all $t \geq 0$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, then a collapsible model is obtained if, and only if, $\mu[t, \theta(t); \beta]$ is linear in $\theta(t)$, i.e.*

$$\mu[t, \theta(t); \beta] = \mu_1[t; \beta_1] + \beta_2\theta(t) \quad \forall t \geq 0,\ \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}. \tag{6}$$

Proof: That (6) implies a collapsible model is trivial by integrating the equation with respect to t. In the other direction, if $\mu[t, \boldsymbol{\theta}_t] \equiv \mu[t, \theta(t)]\ \forall t \geq 0,\ \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$, then $\mu_1[t, y(t)] = \mu_1[t]$ and $\mu_2[t, y(t)] = \mu_2[t]$. From Theorem 3.1, $\frac{\partial}{\partial y}\mu_1[x, y] = \frac{\partial}{\partial x}\mu_2[x, y]$, which implies that $\frac{d}{dt}\ \mu_2[t] = 0$ and thus $\mu_2[t] = \beta_2$. □

Remark: Corollary 3.1 can be proved without reference to Theorem 3.1 by appealing to the strict inequality version of Jensen's inequality (see Durrett, 1991, p. 6, result (3.2)) to show that the collapsible model cannot hold if $\mu[t, \theta(t); \beta]$ is not linear in its second argument.

Hence, if the increment in internal wear at time t depends only on t and the usage rate at that time, the only possible collapsible model that can arise is the linear collapsible model. We can, however, get more general models if we let μ in (4) also depend on $y(t)$ and/or other features of the usage accumulation history $\boldsymbol{\theta}_t$. For example, we can get the log-linear model with $\phi(t, y(t); \beta) = t^{\beta_0} y(t)^{\beta_1}$ by letting $\mu[t, \boldsymbol{\theta}_t; \beta] = \beta_0 t^{\beta_0 - 1} y(t)^{\beta_1} + \beta_1 t^{\beta_0} y(t)^{\beta_1 - 1} \theta(t)$.

We now consider the case where the function μ depends also on $x(t)$. That is, we assume that $x(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s, x(s)]\, ds$. In this case, the solution $x(t)$ is defined only implicitly, via a differential equation, so that more care must be taken. To ensure the existence of a unique solution to the differential equation, we need a mild boundedness condition on μ:

BC $\mu : [0, \infty) \times \boldsymbol{\Theta} \times [0, \infty) \to [0, \infty)$ is such that for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and all $0 \le t_1 < t_2 < \infty$, there exists $\epsilon > 0$ such that

$$\inf_{t_1 \le t \le t_2} \inf_{r \in [0,\infty)} \sup_{b > 0} \frac{b}{\sup_{\substack{t \le s \le t+\epsilon \\ r-b \le z \le r+b}} \mu[s, \boldsymbol{\theta}_s, z]} \ \ge\ \epsilon\,.$$

For example, this condition is trivially satisfied if the function μ is bounded above.

We begin with a lemma.

Lemma 3.4 *Suppose $x(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s, x(s)]\, ds$ for $t \ge 0$, where μ satisfies the* **BC** *condition. Fix $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and let $g_{\boldsymbol{\theta}} : [0, \infty)^2 \to [0, \infty)$ be $g_{\boldsymbol{\theta}}(s, z) = \mu[s, \boldsymbol{\theta}_s, z]$. Let $0 \le t_1 < t_2 < \infty$, and let $x(t_1)$ be given. Assume there is a continuously differentiable function $a : [t_1, t_2] \times [0, \infty) \to [0, \infty)$ such that $g_{\boldsymbol{\theta}}(s, z) = a(s, z)$ for $(s, z) \in (t_1, t_2) \times [0, \infty)$. Then the function $\{x(t)\}_{t \in [t_1, t_2]}$ is uniquely determined, and can be written as $x(t) = x(t_1) + \int_{t_1}^t h_{\boldsymbol{\theta}}(s)\, ds$ for some function $h_{\boldsymbol{\theta}} : [t_1, t_2] \to [0, \infty)$. Here $h_{\boldsymbol{\theta}}(u)$ for $u \in (t_1, t_2)$ depends only on $\boldsymbol{\theta}_u$, and not on $\theta(s)$ for $s > u$. Furthermore, $h_{\boldsymbol{\theta}}(t)$ is continuously differentiable, and $x(t)$ is twice continuously differentiable, on (t_1, t_2).*

Proof: Differentiating, we have that $x'(t) = \mu[t, \boldsymbol{\theta}_t, x(t)] = a(t, x(t))$ for $t \in (t_1, t_2)$. Given the conditions on μ and since a is continuously differentiable, it follows from the standard theory of first-order differential equations (see e.g. Braun, 1983, pp. 76–77) that, given $x(t_1)$, there is a unique solution $\{x(t)\}_{t \in [t_1, t_2]}$ to this equation, and furthermore x is continuous.

In terms of this solution $\{x(t)\}_{t\in[t_1,t_2]}$, we continue as follows. Firstly, since $x'(t) = \mu[t,\boldsymbol{\theta}_t,x(t)] = a(t,x(t))$, with x and a continuous, it follows that x' is continuous, i.e. that x is continuously differentiable. But then, applying this reasoning again, we see that since x and a are continuously differentiable, therefore x' is continuously differentiable, i.e. that x is twice continuously differentiable.

Furthermore, since x is differentiable, we have $x(t) = x(t_1) + \int_{t_1}^{t} x'(s)\,ds$ for $t \in [t_1,t_2]$. We set $h_{\boldsymbol{\theta}}(s) = x'(s)$. Since x' is continuously differentiable, so is $h_{\boldsymbol{\theta}}$.

Finally, since the solution $\{x(t)\}_{t\in[t_1,u]}$ depends only on $a(s,z)$ for $s \le u$, therefore $h_{\boldsymbol{\theta}}(u)$ depends only on θ_s for $s \le u$ and not for $s > u$, as claimed. □

Reformulating this lemma, we obtain the following.

Corollary 3.2 *Suppose $x(t) = \int_0^t \mu[s,\boldsymbol{\theta}_s,x(s)]\,ds$ for $t \ge 0$, where $\mu : [0,\infty)\times\boldsymbol{\Theta}\times[0,\infty) \to [0,\infty)$ satisfies the* **BC** *condition. Let $0 \le t_1 < t_2 < \infty$, and let $x(t_1)$ be given. Let $\boldsymbol{\Theta}_{t_1,t_2}$ be the set of all elements of $\boldsymbol{\Theta}$ which are continuously differentiable on $[t_1,t_2]$. Then there is $\mu^* : [t_1,t_2]\times\boldsymbol{\Theta}_{t_1,t_2} \to [0,\infty)$ such that $x(t) = x(t_1) + \int_{t_1}^{t} \mu^*[s,\boldsymbol{\theta}_s]\,ds$ for $t \in [t_1,t_2]$. Furthermore, the mapping $s \mapsto \mu^*[s,\boldsymbol{\theta}_s]$ is continuously differentiable on (t_1,t_2) for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{t_1,t_2}$. (That is, μ^* is regular when restricted to (t_1,t_2).) Finally, the function $\mu^*[s,\boldsymbol{\theta}_s]$ does not depend on t_1 or t_2; that is, we would obtain the same function μ^* if we started instead with t_1' and t_2', where $t_1 < t_1' < t < t_2' < t_2$.*

Proof: Simply set $\mu^*[s,\boldsymbol{\theta}_s] = h_{\boldsymbol{\theta}}(s)$, for $s \in [t_1,t_2]$.

For the final statement, once we have μ^*, then by the Fundamental Theorem of Calculus we must have $x'(t) = \mu^*[t,\boldsymbol{\theta}_t]$, which does not depend on t_1 and t_2. □

Since the function μ^* does not depend on t_1 or t_2, we can define $\mu^*[t,\boldsymbol{\theta}_t]$ for all $t \ge 0$ at once, to obtain the following.

Corollary 3.3 *Suppose $x(t) = \int_0^t \mu[s,\boldsymbol{\theta}_s,x(s)]\,ds$ for $t \ge 0$, where $\mu : [0,\infty)\times\boldsymbol{\Theta}\times[0,\infty) \to [0,\infty)$ satisfies the* **BC** *condition. Then there is $\mu^* : [0,\infty)\times\boldsymbol{\Theta} \to [0,\infty)$ such that $x(t) = \int_0^t \mu^*[s,\boldsymbol{\theta}_s]\,ds$ for all $t \ge 0$. Furthermore, μ^* is regular.*

Combining the above corollary with Lemma 3.3, we immediately obtain the following.

Theorem 3.2 *Suppose $x(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s, x(s)]\, ds$ for $t \geq 0$, where $\mu : [0, \infty) \times \boldsymbol{\Theta} \times [0, \infty) \to [0, \infty)$ satisfies the* **BC** *condition. Suppose further that there is a function $f : [0, \infty)^2 \to [0, \infty)$ such that $x(t) = f(t, y(t))$ for all $t \geq 0$ and all usage histories $\boldsymbol{\theta}$. Then there exist functions $\mu_1, \mu_2 : [0, \infty)^2 \to [0, \infty)$ which are continuously differentiable, with $\frac{\partial}{\partial y}\mu_1(x, y) = \frac{\partial}{\partial x}\mu_2(x, y)$ for all $x, y \in [0, \infty)^2$, such that for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, we have*

$$\mu[t, \boldsymbol{\theta}_t, x(t)] = \mu_1[t, y(t)] + \mu_2[t, y(t)]\, \theta(t)$$

at all points $t \geq 0$ where $\theta(t)$ is smooth.

Combining Theorem 3.2 with Lemmas 3.2 and 3.4, we obtain

Theorem 3.3 *Suppose that wear is accumulated according to equation (4), where μ satisfies the* **BC** *condition. Then we have a collapsible model if, and only if, there exist functions $\mu_1, \mu_2 : [0, \infty)^2 \to [0, \infty)$ which are continuously differentiable, with $\frac{\partial}{\partial y}\mu_1(x, y) = \frac{\partial}{\partial x}\mu_2(x, y)$ for all $x, y \geq 0$, such that for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,*

$$\mu[t, \boldsymbol{\theta}_t, x(t)] = \mu_1[t, y(t)] + \mu_2[t, y(t)]\, \theta(t)$$

at all times $t \geq 0$ where $\theta(t)$ is smooth.

We conclude this section with the following result, stating that when internal wear is deterministic, the collapsible model is just a special case of the AFT model.

Theorem 3.4 *In the case where internal wear is accumulated according to equation (4), where μ satisfies the* **BC** *condition, the collapsible models are the subset of the accelerated failure time models with $G[\cdot]$ in (1) given by $G[x] = F_{X^*}[x] \equiv P[X^* > x]$, and $\psi(\boldsymbol{\theta}_u; \beta)$ in (1) given by $\psi(\boldsymbol{\theta}_u; \beta) = \mu_1[u, y(u)] + \mu_2[u, y(u)]\theta(u)$ for continuously differentiable functions $\mu_1,\ \mu_2$ with $\partial\mu_1(x, y)/\partial y = \partial\mu_2(x, y)/\partial x$.*

Proof: Simply combine the conclusions of Theorems 3.3 and 2.1. □

4 Stochastic wear

We now consider the more realistic case where the internal wear is a stochastic process that can be described by a stochastic differential equation of the form

$$dX(t) = \mu[t, \boldsymbol{\theta}_t, X(t)]dt + \sigma[t, \boldsymbol{\theta}_t, X(t)]d\gamma(t), \tag{7}$$

with initial condition $X(0) = 0$ with probability 1.

Theorem 4.1 *For any fixed $x^* > 0$, the events $\{T > t\}$ and $\{X(t) < x^*\}$ are equivalent w.p. 1 if and only if $X(t)$ is non-decreasing in t $\forall t \geq 0$ w.p. 1.*

Proof: First, recall that $T = \inf\{u : X(u) \geq x^*\}$. That $X(t)$ non-decreasing in t $\forall t \geq 0$ w.p. 1 implies $\{T > t\} \Leftrightarrow \{X(t) < x^*\}$ $\forall x^* > 0$ w.p. 1 is obvious from the plot of any non-decreasing process $\{X(t), t \geq 0\}$. In the other direction, suppose that there exist $s < t$ and $x^* > 0$ such that $X(s) > x^*$ and $X(t) < x^*$. Then $X(\cdot)$ must be decreasing between s and t. □

Since μ and σ are assumed to be non-negative, assuming that the process $\{\gamma(t); t \geq 0\}$ has non-negative increments is sufficient to guarantee that $X(t)$ is non-decreasing – and we shall assume this henceforth. Theorem 4.1 suggests that, for the most part, collapsible models (or even the AFT) cannot be obtained when $\gamma(t)$ in (7) is, say, a Brownian motion, or any process that allows, with positive probability, paths that are not entirely non-decreasing.

We now look at specific conditions on μ and σ in (7) in order to have a collapsible model. Because of Theorem 2.1, we can assume, without loss of generality, that the threshold is fixed. Before we obtain the main result of the section, we need the following lemma:

Lemma 4.1 *Suppose that wear is accumulated according to equation (7) with $\gamma(t)$ satisfying the following condition:*

C1 $P[\gamma(t) - \gamma(s) \geq 0] = 1, \quad 0 \leq s < t;$

Suppose further that $\mu[\ ]$ and $\sigma[\ ]$ satisfy:

C2 $\mu[t, \boldsymbol{\theta}_t, X(t)] = \mu[t, \boldsymbol{\theta}_t]$;

C3 $\sigma[t, \boldsymbol{\theta}_t, X(t)] = \sigma[t]$.

Then $P[T > t|\boldsymbol{\theta}]$ *depends on* $\boldsymbol{\theta}$ *only through* t *and* $y(t)$, *if and only if* $\int_0^t \mu[s, \boldsymbol{\theta}_s]ds$ *is a function of* t *and* $y(t)$ *only,*

Proof: Under condition **C1**, Theorem 4.1 implies that $P[T > t|\boldsymbol{\theta}] = P[X(t) < x^*|\boldsymbol{\theta}]$. Hence,

$$\begin{aligned} P[T > t|\boldsymbol{\theta}] &= P[X(t) < x^*|\boldsymbol{\theta}] = P\left[\int_0^t \mu[s, \boldsymbol{\theta}_s]ds + \int_0^t \sigma[s]d\gamma(s) < x^*\right] \\ &= P\left[\int_0^t \sigma[s]d\gamma(s) < x^* - \int_0^t \mu[s, \boldsymbol{\theta}_s]ds\right] \\ &= F_{\gamma^*(t)}\left[x^* - \int_0^t \mu[s, \boldsymbol{\theta}_s]ds\right], \end{aligned}$$

where $F_{\gamma^*(t)}$ is the left-continuous c.d.f. of the process $\{\gamma^*(s),\ s \geq 0\}$ at $s = t$ given by $\gamma^*(t) = \int_0^t \sigma[s]d\gamma(s)$. Hence, $P[T > t|\boldsymbol{\theta}]$ depends on $\boldsymbol{\theta}$ only through t and $\int_0^t \mu[s, \boldsymbol{\theta}_s]ds$. The result follows. □

Theorem 4.2 *Suppose that wear is accumulated as in equation (7), with* $\gamma(t)$ *satisfying condition* **C1** *and* $\mu[\ ]$ *and* $\sigma[\ ]$ *satisfying conditions* **C2**-**C3** *of Lemma 4.1, and with* $\mu[\ ]$ *regular. Then we have a collapsible model if, and only if, there exist continuously differentiable functions* $\mu_1, \mu_2 : [0, \infty)^2 \to [0, \infty)$ *with* $\partial\mu_1(x, y)/\partial y = \partial\mu_2(x, y)/\partial x$ *for* $x, y \geq 0$, *such that* $\mu[t, \boldsymbol{\theta}_t] = \mu_1[t, y(t)] + \mu_2[t, y(t)]\theta(t)$ *for all* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ *and all* $t \geq 0$.

Proof: Direct consequence of Lemma 4.1 and Theorem 3.1. □

Hence, we see that a collapsible model can arise from a model where failure is caused by the internal wear crossing a threshold, with internal wear being a process of the form (7) with $\mu[t, \boldsymbol{\theta}_t]$ regular, $\sigma[t, \boldsymbol{\theta}_t, X(t)] = \sigma[t]$ and $\gamma(t)$ being a process with non-negative increments, such as the gamma process (see Wenocur, 1989 or Singpurwalla, 1995, for example).

It seems that the conditions of Theorem 4.2 are essentially the only way to obtain a collapsible model in this case. However, it appears difficult to formulate this into a precise theorem.

Since semiparametric modeling is possible (i.e., we parametrically specify $\phi(t, y(t); \beta)$ but leave $G(\cdot)$ arbitrary in (3)), it is not necessary for $F_{\gamma^*(t)}$ to be mathematically tractable.

5 Presence of traumatic events

We now use a different approach to model the relationship between internal wear and failure. Cox (1999, Section 3) describes how internal wear can be *rate determining*, i.e. $\{X(t),\ t \geq 0\}$ affects time of failure, K, by determining the hazard function

$$\lambda(t|X(s), 0 \leq s \leq t) = \lim_{h \downarrow 0} \frac{P[K \in [t, t+h)|K \geq t, X(s), 0 \leq s \leq t]}{h}. \tag{8}$$

Many authors (e.g. Jewell and Kalbfleisch, 1996; Cox, 1999; Bagdonavičius and Nikulin, 2001) have considered the additive hazards model as a potential specification of (8) in related frameworks:

$$\lambda(t|X(s), 0 \leq s \leq t) = \lambda_0 + \beta X(t), \tag{9}$$

where λ_0 and β are such that the probability of $\lambda(t|X)$ taking on negative values is negligible.

Singpurwalla (1995) and Bagdonavičius and Nikulin (2001) generalize this idea by considering two competing causes of failure: internal wear reaching a threshold and occurrence of a traumatic event (accident) that "kills" the item. The hazard of occurrence of a traumatic event is modelled as in Cox (1999), i.e. the hazard function of a traumatic event at time t depends on the value of the internal wear process at that time. Bagdonavičius and Nikulin (2001) consider the case when covariates (such as usage history) are available. In that case, let K be the time at which a traumatic event happens. Then the hazard of a traumatic event at time t is given by

$$\lambda(t \mid X(s), \theta(s), 0 \leq s \leq t) = \lim_{h \downarrow 0} \frac{P[K \in [t, t+h)|K \geq t, X(s), \theta(s), 0 \leq s \leq t]}{h}. \tag{10}$$

The time to failure random variable in this context is $U = \min(T, K)$, where T is the time at which the internal wear process crosses the failure threshold (as considered in previous sections).

In the remainder of this section, we first consider what type of hazard function (8) is compatible with collapsible models when only traumatic events are possible. We then look at the more general case allowing for failure to be caused by both excessive internal wear and traumatic events.

5.1 Failure caused by traumatic events

Suppose that failure can only be caused by the occurrence of a traumatic event, and let K be the time at which such a traumatic event happens. We are thus interested in the conditional survivor function

$$\begin{aligned} G_K(t) &= P[K > t|\boldsymbol{\theta}_t] \\ &= E\Big[P[K > t|X(s), \theta(s), 0 \le s \le t]\Big|\boldsymbol{\theta}_t\Big] \\ &= E\left[\exp\left\{-\int_0^t \lambda(s|X(v), 0 \le v \le s)ds\right\}\middle|\boldsymbol{\theta}_t\right]. \end{aligned} \tag{11}$$

5.1.1 Deterministic wear

If internal wear is a deterministic function of usage and time, i.e. $x(t)$ is specified by the differential equation (4) and is entirely determined by $\boldsymbol{\theta}$, then (11) can be simplified to

$$\begin{aligned} G_K(t) &= \exp\left\{-\int_0^t \lambda(s|x(v), 0 \le v \le s)ds\right\} \qquad (12) \\ &= \exp\left\{-\int_0^t \lambda^*(s|\boldsymbol{\theta}_s)ds\right\}, \end{aligned}$$

where $\lambda^*(s|\boldsymbol{\theta}_s) = \lim_{h\downarrow 0} P[K \in [t, t+h)|K \ge t, \boldsymbol{\theta}_t]/h$. Because $\exp\{-\cdot\}$ is a strictly decreasing survivor function, we are in the presence of the accelerated failure time model, and we have the following.

Theorem 5.1 *In an environment where failure can only be caused by traumatic events, and the hazard of such events at any given time is a function of an internal wear that is itself a deterministic function of the usage history, then provided λ^* is regular, the conditional survivor function $G_K(t)$ is of the collapsible form if, and only if,*

$$\lambda^*(t|\boldsymbol{\theta}_t) = \lambda_1(t, y(t)) + \lambda_2(t, y(t))\theta(t) \quad \forall t \ge 0,\ \boldsymbol{\theta} \in \boldsymbol{\Theta}$$

for some continuously differentiable functions λ_1 *and* λ_2 *such that* $\frac{\partial}{\partial y}\lambda_1(x,y) = \frac{\partial}{\partial x}\lambda_2(x,y)$ *for all* $x, y \in [0,\infty)$.

Proof: Same arguments as for Theorem 3.1. □

We now consider two special cases of interest. First, suppose that the conditional hazard of a traumatic event at time t only depends on time and the usage rate at that time, $\theta(t)$. One interpretation of this is that accidents that "kill" the item are more likely when usage is more intense. Under these conditions, we obtain the following result:

Corollary 5.1 *If* $\lambda^*(t|\boldsymbol{\theta}_t) \equiv \lambda^*(t|\theta(t))$, *then* $G_K(t)$ *will be of the collapsible form if and only if*

$$\lambda^*(t|\theta(t)) = \lambda_0(t) + \beta\theta(t).$$

Proof: Direct consequence of Theorem 5.1, just like the proof of Corollary 3.1. □

The second special case considered is the additive hazard model (9) based on the value of $x(t)$, i.e.

$$\lambda(t|x(t)) = \lambda_0(t) + \beta x(t) = \lambda_0(t) + \beta \int_0^t \mu[s, x(s), \boldsymbol{\theta}_s] ds.$$

If μ only depends on s and $\boldsymbol{\theta}_s$, then equation (12) implies that we have a collapsible model if, and only if, there exists a positive $[0,\infty)^2 \to [0,\infty)$ function f such that

$$\int_0^t \int_0^s \mu[v, \boldsymbol{\theta}_v]\, dv\, ds = f(t, y(t)), \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\ t \geq 0. \tag{13}$$

A double application of the Fundamental Theorem of Calculus yields that the function $\mu[v, \boldsymbol{\theta}_v]$ in (13) must depend explicitly on $\theta'(v)$, the derivative of the usage rate at time v. Thus it is possible to have a collapsible model in this case, but the fact that μ must explicitly depend on θ' makes those models somewhat unnatural.

5.1.2 Stochastic wear

If wear is a stochastic function of time, i.e. is accumulated according to the differential equation (7), then equation (11) does not simplify as easily. Following Jewell and Kalbfleisch

(1996), Cox (1999), and Bagdonavičius and Nikulin (2001), we consider the case where $\lambda(t|X(s), 0 \leq s \leq t) = \lambda_0(t) + \beta X(t)$. Straightforward calculation (see Jewell and Kalbfleisch, 1996 or Cox, 1999) yields

$$G_K(t) = \exp\left\{-\Lambda_0(t)\right\} E\left[\exp\left\{-\beta \int_0^t X(s)ds\right\} \middle| \boldsymbol{\theta}_t\right], \tag{14}$$

where $\Lambda_0(t) = \int_0^t \lambda_0(s)ds$.

Theorem 5.2 *Suppose that $\mu[\cdot]$, $\sigma[\cdot]$ and $\gamma[\cdot]$ are as in Theorem 4.2. Then we have a collapsible model if, and only if, there exists a function f such that (13) holds.*

Proof: With $\{X(t), t \geq 0\}$ given by (7), and under the hypotheses of Theorem 4.2, (14) simplifies to

$$\begin{aligned} G_K(t) &= \exp\left\{-\Lambda_0(t)\right\} E\left[\exp\left\{-\beta \int_0^t \left(\int_0^s \mu[v, \boldsymbol{\theta}_v]dv \right.\right.\right. \\ &\qquad \left.\left.\left. + \int_0^s \sigma[v]d\gamma[v]\right) ds\right\} \middle| \boldsymbol{\theta}_t\right] \\ &= e^{-\Lambda_0(t)} \exp\left\{-\beta \int_0^t \int_0^s \mu[v, \boldsymbol{\theta}_v]dv\, ds\right\} E\left[e^{-\beta\gamma^{**}(t)}\right], \end{aligned} \tag{15}$$

where $\gamma^{**}(t) = \int_0^t \int_0^s \sigma[v]d\gamma[v]dv\, ds$. Since $\sigma[\cdot]$ does not depend on $\boldsymbol{\theta}$, the expectation in (15) will not depend on the usage history and can be viewed as a (completely specified) deterministic function of t. Hence, the survivor function will be of the collapsible form if and only if $\int_0^t \int_0^s \mu[v, \boldsymbol{\theta}_v]dv\, ds$ is a function of t and $y(t)$. □

Hence, combining Theorem 5.2 with the discussion following equation (13), we see that collapsible models are possible under this framework, but only with a somewhat unnatural μ that must depend explicitly on the derivative of the usage rate.

5.2 Two causes of failure

We now consider the case where failure happens at the occurrence of the first of a traumatic event and internal wear crossing a threshold. The survivor function we wish to model is thus $G_U(t) = P[U > t|\theta_t]$, where $U = \min(K, T)$. Without loss of generality (apply the arguments of the proof of Theorem 2.1 to equation (16) below), we assume that the threshold $X^* = x^*$, a positive constant, w.p. 1. From Bagdonavičius and Nikulin (2001), we have that

$$G_U(t) = E\Big[\exp\Big\{-\int_0^t \lambda(s|X(v), \theta(v), 0 \leq v \leq s)ds\Big\} \times I[X(t) < x^*]\Big|\boldsymbol{\theta}_t\Big]. \tag{16}$$

5.2.1 Determinisitic wear

Let us first look at the case where $X(t)$ is a deterministic function of usage and time, i.e. $x(t)$ is specified through equation (4). In that case, (16) simplifies to

$$\begin{aligned} G_U(t) &= \exp\Big\{-\int_0^t \lambda(s|x(v), \theta(v), 0 \leq v \leq s)ds\Big\} I[x(t) < x^*] \\ &= \exp\Big\{-\int_0^t \lambda^*(s|\boldsymbol{\theta}_s)ds\Big\} I[x(t) < x^*]. \end{aligned} \tag{17}$$

For the model to be collapsible, $G_U(t)$ must be a function of t and $y(t)$ only for each t, $\boldsymbol{\theta}$ and x^*. Hence, letting $x^* \to \infty$ in (17), we see that the integral of λ^* must be a function of t and $y(t)$ only. This, in turns, implies that $I[x(t) < x^*]$ must also be a function of t and $y(t)$. Combining this with the results of Section 3.1 and Section 5.1, we obtain the following.

Theorem 5.3 *Suppose that failure happens at the occurrence of the first of internal wear crossing a threshold and a traumatic event. Suppose that internal wear is accumulated according to the deterministic differential equation (4) and that the hazard of a traumatic event is given by (10). Then the collapsible model is achieved if, and only if, all of the following conditions are met:*

1. *the conditions on $\mu[t, \boldsymbol{\theta}_t, x(t)]$ given in Theorem 3.3;*
2. *the conditions on $\lambda^*(t|\boldsymbol{\theta}_t)$ given in Theorem 5.1.*

5.2.2 Stochastic wear

For the stochastic case where $X(t)$ is defined according to equation (7), let us again consider the additive hazards model (9). From equations (15) and (16) we obtain

$$\begin{aligned} G_U(t) &= e^{-\Lambda_0(t)} \exp\Big\{-\beta \int_0^t \int_0^s \mu[v, \boldsymbol{\theta}_v]dvds\Big\} \\ &\quad \times E\left[e^{-\gamma^{**}(t)} I\left[\gamma^*(t) < x^* - \int_0^t \mu[s, \boldsymbol{\theta}_s]ds\right] \;\Big|\; \boldsymbol{\theta}_t\right]. \end{aligned} \tag{18}$$

Similarly to the deterministic case, both the double and the single integrals on the right-hand side of (18) have to be functions of t and $y(t)$ only for $G_U(t)$ to be a function of t and $y(t)$ only for each t, $\boldsymbol{\theta}$ and x^*. Hence, we get a collapsible model if, and only if, there exist functions f and g such that $\int_0^t \mu[s, \boldsymbol{\theta}_s]ds = f(t, y(t))$ and $\int_0^t \int_0^s \mu[v, \boldsymbol{\theta}_v]dvds = g(t, y(t))$. This leads to the following result:

Theorem 5.4 *Consider an environment where failure occurs at the first of internal wear reaching a threshold and a traumatic event. Assume the hazard of a traumatic event given by the additive hazard model (9), and internal wear is accumulated as in Theorem 4.2. Then we cannot obtain a collapsible model except in the trivial case where $\mu[t, \boldsymbol{\theta}_t]$ is a function of t alone.*

Proof: Suppose the conditions on μ and $\boldsymbol{\theta}$ given in Theorem 3.1. Suppose there exist functions $f,\ g,\ [0,\infty)^2 \to [0,\infty)$ such that for all $t \geq 0,\ \boldsymbol{\theta} \in \boldsymbol{\Theta}$,

$$\int_0^t \mu[s, \boldsymbol{\theta}_s]ds = f(t, y(t)) \tag{19}$$

$$\int_0^t \int_0^s \mu[v, \boldsymbol{\theta}_v]dvds = g(t, y(t)). \tag{20}$$

Theorem 3.1 and equation (19) imply that

$$\mu[s, \boldsymbol{\theta}_s] = f^{(1)}[s, y(s)] + f^{(2)}[s, y(s)]\theta(s) \tag{21}$$

with $\frac{\partial}{\partial y} f^{(1)}(x, y) = \frac{\partial}{\partial x} f^{(2)}(x, y)$. Similarly, Theorem 3.1 and equation (20) imply that

$$\int_0^s \mu[v, \boldsymbol{\theta}_v]dv = g^{(1)}[s, y(s)] + g^{(2)}[s, y(s)]\theta(s) \tag{22}$$

with $\frac{\partial}{\partial y} g^{(1)}(x, y) = \frac{\partial}{\partial x} g^{(2)}(x, y)$. Substituting (21) into (22) we get that

$$\begin{aligned} f(s, y(s)) &= \int_0^s \left(f^{(1)}[v, y(v)] + f^{(2)}[v, y(v)]\theta(v)\right) dv \\ &= g^{(1)}[s, y(s)] + g^{(2)}[s, y(s)]\theta(s) \end{aligned}$$

for all $s \geq 0,\ \boldsymbol{\theta} \in \boldsymbol{\Theta}$. Since $f(s, y(s))$ depends on $\theta(s)$ only through $y(s)$, we must have that $g^{(2)}[s, y(s)] = 0$. This, in turn, implies that $g[t, y(t)] = g^*(t)$, i.e. that g is a function of t alone. A double application of the Fundamental Theorem of Calculus to (20) then shows that $\mu[t, \boldsymbol{\theta}_t]$ is also a function of t alone, as claimed. □

6 Conclusion

We have considered both deterministic and stochastic models for the accumulation of internal wear, given a usage history. For both of these models, we have derived conditions under which collapsible models can arise. Table 1 summarizes our results, obtained under mild regularity conditions as discussed in the paper.

We did not cover the cases in which the effect of usage is modelled through a change in the time-scale of the internal wear process. Such an approach would no doubt lead to other setups in which collapsible models are plausible. We also did not consider the case where cumulative usage is the result of a counting process, such as the cumulative number of startups of a machine. When the number of observed jumps is large and the process can be approximated by a differentiable function, such as the cumulative number of loading cycles of aluminum specimen in Section 7 of Kordonsky and Gertsbakh (1997), then it is reasonable to think that the results derived in this paper would still hold. However, when the number of jumps is small, a new approach to this problem must be taken. We hope to investigate these issues in future work.

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Deterministic wear, i.e., $x(t) = \int_0^t \mu[s, \theta_s, x(s)]\, ds$	
E.W.O.	• $\mu[s, \theta_s, x(s)] = \mu^{(1)}[s, y(s)] + \mu^{(2)}[s, y(s)]\, \theta(s)$ with $\frac{\partial}{\partial y}\mu^{(1)}(x, y) = \frac{\partial}{\partial x}\mu^{(2)}(x, y)$. • If $\mu[s, \theta_s, x(s)] = \mu[s, \theta(s)]$, then $\mu[s, \theta_s] = \mu^{(1)}(s) + \beta\theta(s)$.
T.E.O.	• $\lambda^*(s\vert\theta_s) = \lambda^{(1)}(s, y(s)) + \lambda^{(2)}(s, y(s))\, \theta(s)$, with $\frac{\partial}{\partial y}\lambda^{(1)}(x, y) = \frac{\partial}{\partial x}\lambda^{(2)}(x, y)$. • If $\lambda^*(s\vert\theta_s) = \lambda^*(s\vert\theta(s))$, then $\lambda(s\vert\theta(s)) = \lambda^{(1)}(s) + \beta\theta(s)$. • If $\lambda^*(s\vert x(s)) = \lambda_0(s) + \beta x(s)$, then $\int_0^t \int_0^s \mu[v, \theta_v, x(v)]\, dv\, ds = f(t, y(t))$.
E.W. & T.E.	• $\mu[s, \theta_s, x(s)] = \mu^{(1)}[s, y(s)] + \mu^{(2)}[s, y(s)]\, \theta(s)$, with $\frac{\partial}{\partial y}\mu^{(1)}(x, y) = \frac{\partial}{\partial x}\mu^{(2)}(x, y)$. AND • $\lambda^*(s\vert\theta_s) = \lambda^{(1)}(s, y(s)) + \lambda^{(2)}(s, y(s))\, \theta_s$, with $\frac{\partial}{\partial y}\lambda^{(1)}(x, y) = \frac{\partial}{\partial x}\lambda^{(2)}(x, y)$.

Stochastic wear, i.e., $X(t) = \int_0^t \mu[s, \theta_s, X(s)]\, ds + \int_0^t \sigma[s, \theta_s, X(s)]\, d\gamma(s)$	
E.W.O.	• $\mu[s, \theta_s, X(s)] = \mu^{(1)}[s, y(s)] + \mu^{(2)}[s, y(s)]\, \theta(s)$ with $\frac{\partial}{\partial y}\mu^{(1)}(x, y) = \frac{\partial}{\partial x}\mu^{(2)}(x, y)$. AND • $\sigma[s, \theta_s, X(s)] = \sigma(s)$
T.E.O.	• If $\lambda[s, X(s)] = \lambda_0(s) + \beta X(s)$, then $\int_0^t \int_0^s \mu[v, \theta_v]\, dv\, ds = f(t, y(t))$ AND $\sigma[s, \theta_s, X(s)] = \sigma(s)$.
E.W. & T.E.	• If $\lambda[s, X(s)] = \lambda_0(s) + \beta X(s)$, and $\sigma[s, \theta_s, X(s)] = \sigma(s)$, then must have $\mu[s, \theta_s, x(s)] = \mu(s)$.

Table 1: Summary of the conditions required to get simple collapsible models in various setups. “E.W.O.” stands for failure due to excessive wear only; “T.E.O.” stands for failure due to traumatic event only; and “E.W. & T.E.” stands for failure due to the earlier of excessive wear and traumatic event.