A Cancellativity Criterion for a Class of Spin Systems

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Abstract

How much noise must be added to a voter model (in the sense of Liggett (1997)) to render it ergodic? A partial answer is obtained by studying algebraic conditions on the flip rates associated with cancellative duality. It is shown that spontaneous births and deaths must both attain a certain threshold for cancellative duality to occur. The threshold is calculated for the nonlinear voter models of Cox and Durrett (1991), and the result leads to lower bounds on the critical parameter values for Ising models.

Keywords: Voter model; Noisy voter model; Ergodicity; Cancellative duality; Ising model

1 Prologue

In a basic version of the voter model (Holley and Liggett (1975); see also Clifford and Sudbury (1973)), each point of a finite or countable set S (usually Z^d) is thought of as the site of a "voter" with one of two affiliations, 0 or 1. Each $x \in S$ has a finite set of neighbouring sites, and when the system is in configuration $\eta \in \{0,1\}^S$, the voter at x tends to change affiliation at a rate $c(x, \eta)$ equal to the number of neighbours of the opposite affiliation. Cox and Durrett (1991) introduced nonlinear voter models, for which the flip rate $c(x, \eta)$ is an arbitrary function f of the number of disagreeing neighbours, subject only to the condition $f(0) = 0$. In his 1996 Wald lectures, Liggett

(1997, §4) broadened the definition of a voter model to include any system of 0's and 1's flipping individually at rates determined by their neighbours which satisfies two conditions:

- ($0 \leftrightarrow 1$ invariance) If $\zeta(y) = 1 - \eta(y)$ for all $y \in S$ then $c(x,\eta) = c(x,\zeta)$ for each $x \in S$.
- (No spontaneous births or deaths) $c(x,\eta) = 0$ for all $x \in S$ if $\eta \equiv 0$ or $\eta \equiv 1$.

Granovsky and Madras (1995, p. 32) observed that adding any noise, i.e. flips of affiliation in the absence of neighbours of the opposite camp, renders the first, "classical" voter model described above exponentially ergodic (see Theorem I.4.1 and inequality III.0.6 in Liggett (1985)). On the other hand, a large class of spin systems can be obtained by adding noise to voter models of the broader, more recently studied type; and by no means all such "noisy voter models" are ergodic. Intuitively, the more noise, the greater the tendency toward ergodicity. It seems natural, then, to ask how much noise will suffice to turn a (generalized) voter model—which by its second defining property has at least two invariant distributions—into an ergodic system with just one.

Formally, let us define a noisy voter model (NVM) as a spin system whose flip rates have the form

$$c'(x,\eta) = \begin{cases} c(x,\eta) + \beta_x & \text{if } \eta(x) = 0 \\ c(x,\eta) + \delta_x & \text{if } \eta(x) = 1 \end{cases} \tag{1}$$

for some nonnegative β_x, δ_x, where the $c(x,\eta)$ are flip rates of some voter model. (The letter β stands for birth, while δ stands for death—as becomes more appropriate if we think of the system as one of particles being created and destroyed.) This notion of NVM's generalizes both the voter models with defections of Durrett (1981, pp. 124-6) and the noisy voter models of Granovsky and Madras (1995).

As already noted, a given NVM may or may not be ergodic: for instance, all Ising models, ergodic and nonergodic, are NVM's. Finding a useful characterization of the ergodic NVM's may be an intractable problem. The theorem proved here answers the easier question of which NVM's are cancellative (see below), then uses the answer to find a sufficient condition for ergodicity.

2 Statement and proof of the criterion

Before stating our theorem we need some notation and a definition.

1. For $\eta \in \{0,1\}^S$ and finite $F \subset S$, define

$$\chi_F(\eta) = (-1)^{|\eta \cap F|}$$

where $\eta \cap F$ denotes the set of sites in F which are occupied in η. (χ_F is essentially χ_F^α as defined in Holley and Stroock (1979), in the case $\alpha \equiv 0$.)

2. For $x \in S$ let $\mathcal{N}_x$ comprise x and its neighbours. Throughout this paper $\mathcal{N}_x$ is assumed finite.

3. We call a spin system on S cancellative if for each $x \in S$ and $\eta \in \{0,1\}^S$,

$$c(x,\eta) = \frac{a_x}{2}[1 - \sum_{F \subset \mathcal{N}_x} q(x,F)\chi_{F \triangle \{x\}}(\eta)] \tag{2}$$

for some constants a_x, $q(x,F)$ such that the $p(x,F) \equiv |q(x,F)|$ satisfy

$$\sum_{F \subset \mathcal{N}_x} p(x,F) \leq 1. \tag{3}$$

A cancellative system is one with a cancellative (or annihilating) dual process in reverse time. Conditions (2) and (3) together are necessary and sufficient for the existence of such a dual. See Liggett (1985, §III.4) for details. An alternative approach to duality, based upon a graphical representation, is developed in Griffeath (1979). More recently, Sudbury and Lloyd (1995) have presented an algebraic formulation of duality employing quantum operators.

We now state our main result.

Theorem. **(a)** *A noisy voter model is cancellative iff*

$$min\{\beta_x, \delta_x\} \geq T_x(c) \text{ for all } x \in S,$$

where $T_x(c)$ is a number determined by the function $c(x,\cdot)$.

(b) *Under the stronger hypothesis*

$$inf_{x\in S}[min\{\beta_x, \delta_x\} - T_x(c)] > 0,$$

the system is ergodic.

Remarks.

1. The values $T_x(c)$ may be thought of as noise thresholds which must be attained at each x if the process is to have a cancellative dual.
2. Note that this result is independent of the geometry of the neighbourhood. Sharper ergodicity results for NVM's might be obtainable by taking the neighbourhood structure into account (see Belitsky and Granovsky (1991)).

The bulk of this section is devoted to proving the above theorem. Proving part (a) will take some work; (b) will then follow from a result due to Holley and Stroock.

Proof of (a). To make the notation cleaner, fix $x \in S$ and omit the subscripts (e.g., let $a = a_x$). Also let $c(\eta) = c(x, \eta)$, $q(F) = q(x, F)$ and $p(F) = p(x, F)$. Equations (2) and (3) then become

$$c(\eta) = \frac{a}{2}[1 - \sum_{F\subset\mathcal{N}} q(F)\chi_{F\triangle\{x\}}(\eta)], \tag{4}$$

$$\sum_{F\subset\mathcal{N}} p(F) \leq 1. \tag{5}$$

Proving part (a) amounts to finding necessary and sufficient conditions for the existence of constants a, $q(F)$ ($F \subset \mathcal{N}$) which satisfy (4) for each η while obeying (5). We proceed through five steps.

Step 1: Reduction to the $q(x) = 0$ case. It would simplify the algebra if we could assume that $q(x) = 0$ (where $q(x) = q(\{x\})$). Moreover, for each $F \subset \mathcal{N}$, $p(F)$ determines the rate of branchings of the dual process in which a particle at x is replaced by particles throughout F (with annihilation occurring at the already-occupied sites of F; see Liggett (1985), p. 160). In particular $p(x)$ corresponds to null branchings in which x is simply replaced by x. This suggests that, intuitively, $q(x)$ "might as well" be 0.

In fact no generality is lost in assuming $q(x) = 0$, since if a, $q(F)$ satisfy (4), (5), then there exist a', $q'(F)$ (with $p'(F) \equiv |q'(F)|$) satisfying (4′), (5′) (i.e. (4), (5) with appropriate primes added) such that $q'(x) = 0$. To wit, let

$$a' = a(1 - q(x)),$$

$$q'(F) = \frac{q(F)}{1 - q(x)} \text{ for all } F \subset \mathcal{N} \text{ except } \{x\},$$

and (again) $q'(x) = 0$. Then (4′) holds since

$$\begin{aligned}
c(\eta) &= \frac{a}{2}[1 - q(x)\chi_{\emptyset}(\eta) - \sum_{F \neq \{x\}} q(F)\chi_{F \triangle \{x\}}(\eta)] \\
&= \frac{a'}{2(1 - q(x))}[1 - q(x) - \sum_{F \neq \{x\}} q'(F)(1 - q(x))\chi_{F \triangle \{x\}}(\eta)] \\
&= \frac{a'}{2}[1 - \sum_{F \neq \{x\}} q'(F)\chi_{F \triangle \{x\}}(\eta)] \\
&= \frac{a'}{2}[1 - \sum_{F \subset \mathcal{N}} q'(F)\chi_{F \triangle \{x\}}(\eta)],
\end{aligned}$$

and (5′) holds because

$$\begin{aligned}
\sum_{F \subset \mathcal{N}} p'(F) &= \frac{\sum_{F \neq \{x\}} p(F)}{1 - q(x)} \\
&\leq \frac{\sum_{F \neq \{x\}} p(F)}{1 - p(x)} \\
&\leq \frac{\sum_{F \neq \{x\}} p(F)}{\sum_{F \subset \mathcal{N}} p(F) - p(x)} \\
&= 1.
\end{aligned}$$

Thus we may henceforth assume that $q(x) = 0$. This allows us to find unique solutions for $q(F)$ ($F \subset \mathcal{N}$) satisfying (4). The system is then cancellative iff, for each x, these values satisfy $\sum_{F \subset \mathcal{N}} |q(F)| \leq 1$, or the equivalent but easier-to-apply inequality

$$a[1 - \sum_{F \subset \mathcal{N}} |q(F)|] \geq 0. \tag{6}$$

Step 2: Labelling the 2^n configurations of the neighbours and the 2^{n+1} subsets of $\mathcal{N}$. Let $\eta_1, \ldots, \eta_{2^n}$ be configurations such that $\eta_i(x) = 0$ for each i and

$$\eta_1|_{\mathcal{N}\backslash\{x\}}, \ldots, \eta_{2^n}|_{\mathcal{N}\backslash\{x\}}$$

are the 2^n distinct configurations on $\mathcal{N}\backslash\{x\}$. The only stipulation on the numbering is that η_1 be the all-vacant configuration: $\eta_1(y) = 0$ for all $y \in \mathcal{N}$. For each i let ζ_i be the "mirror-opposite" configuration given by $\zeta_i(y) = 1 - \eta_i(y)$ for all $y \in S$. Let E_i (resp. O_i) be the set of even (resp. odd) cardinality such that $E_i\backslash\{x\}$ ($O_i\backslash\{x\}$) is the set of occupied sites in $\mathcal{N}$ of η_i. Note that $E_1 = \emptyset$, $O_1 = \{x\}$. Also let $b_i = c'(\eta_i)$, $d_i = c'(\zeta_i)$ (b for birth since $\eta_i(x) = 0$ for each i, d for death since $\zeta_i(x) = 1$ for each i).

With all this notation in hand, we can solve equation (4) for the $aq(F)$'s ($F \subset \mathcal{N}$), and relate the solution to the cancellativity inequality (6). In the next two steps we solve for the $aq(E_i)$'s and the $aq(O_i)$'s separately.

Step 3: Solving for $aq(E_1), \ldots, aq(E_{2^n})$. One can readily show that

$$\chi_{F\triangle\{x\}} = \begin{cases} \chi_F & \text{if } x \text{ is vacant} \\ -\chi_F & \text{if } x \text{ is occupied.} \end{cases}$$

Using this fact and (4), we have

$$b_i = \frac{a}{2}[1 - \sum_{F\subset\mathcal{N}} q(F)\chi_F(\eta_i)], \tag{7}$$

$$d_i = \frac{a}{2}[1 + \sum_{F\subset\mathcal{N}} q(F)\chi_F(\zeta_i)]. \tag{8}$$

By (7) and (8),

$$\begin{aligned} \delta - \beta &= d_i - b_i \\ &= \frac{a}{2}\sum_{F\subset\mathcal{N}} q(F)[\chi_F(\eta_i) + \chi_F(\zeta_i)] \\ &= \frac{a}{2}[\sum_{j=1}^{2^n}(\chi_{E_j}(\eta_i) + \chi_{E_j}(\zeta_i)) + \sum_{j=1}^{2^n}(\chi_{O_j}(\eta_i) + \chi_{O_j}(\zeta_i))]. \end{aligned}$$

Now,

$$\chi_{E_j}(\eta_i) + \chi_{E_j}(\zeta_i) = (-1)^{|\eta_i\cap E_j|} + (-1)^{|\zeta_i\cap E_j|}$$

$$\begin{aligned}
&= (-1)^{|\eta_i \cap E_j|} + (-1)^{|E_j| - |\eta_i \cap E_j|} \\
&= (-1)^{|\eta_i \cap E_j|} + (-1)^{-|\eta_i \cap E_j|} \text{ (since } |E_j| \equiv 0 \pmod 2) \text{)} \\
&= 2\chi_{E_j}(\eta_i),
\end{aligned}$$

and $\chi_{O_j}(\eta_i) + \chi_{O_j}(\zeta_i) = 0$ by a similar argument. Hence

$$\begin{aligned}
\delta - \beta &= a \sum_{j=1}^{2^n} q(E_j)\chi_{E_j}(\eta_i) \\
&= a \sum_{j=1}^{2^n} q(E_j) m_{ij}
\end{aligned}$$

for each i, where $m_{ij} = \chi_{E_j}(\eta_i) = \chi_{O_j}(\eta_i)$. Letting M be the matrix $(m_{ij})_{1 \le i,j \le 2^n}$, we then have

$$(\delta - \beta) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = aM \begin{pmatrix} q(E_1) \\ \vdots \\ q(E_{2^n}) \end{pmatrix}. \tag{9}$$

At this point we require the following two elementary lemmata about M, which are proved in Section 5.

Lemma 1 $\frac{1}{2^n} M \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Lemma 2 $\frac{1}{2^n} M^2 = I$.

Multiplying both sides of (9) by $\frac{1}{2^n} M$ and applying the lemmata yields

$$(\delta - \beta) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a \begin{pmatrix} q(E_1) \\ q(E_2) \\ \vdots \\ q(E_{2^n}) \end{pmatrix},$$

so that

$$aq(E_1) = \delta - \beta,\ aq(E_i) = 0 \text{ for } i = 2, 3, \ldots, 2^n. \tag{10}$$

Step 4: Solving for $aq(O_1), \ldots, aq(O_{2^n})$. We can now compute, for each i,

$$\begin{aligned}
b_i &= \frac{a}{2}[1 - \sum_{j=1}^{2^n}\{q(E_j)\chi_{E_j}(\eta_i) + q(O_j)\chi_{O_j}(\eta_i)\}] \\
&= \frac{a}{2} - \frac{\delta-\beta}{2} - \frac{a}{2}\sum_{j=1}^{2^n} q(O_j)m_{ij}
\end{aligned}$$

by (10), so that

$$\begin{aligned}
\begin{pmatrix} 2b_1 \\ 2b_2 \\ \vdots \\ 2b_{2^n} \end{pmatrix} &= \begin{pmatrix} a-\delta+\beta - a\sum_{j=1}^{2^n} q(O_j)m_{1,j} \\ a-\delta+\beta - a\sum_{j=1}^{2^n} q(O_j)m_{2,j} \\ \vdots \\ a-\delta+\beta - a\sum_{j=1}^{2^n} q(O_j)m_{2^n,j} \end{pmatrix} \\
&= (a-\delta+\beta)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - aM\begin{pmatrix} q(O_1) \\ q(O_2) \\ \vdots \\ q(O_{2^n}) \end{pmatrix}.
\end{aligned}$$

As above we multiply by $\frac{1}{2^n}M$ and apply the lemmata; the result is

$$\begin{pmatrix} aq(O_1) \\ aq(O_2) \\ \vdots \\ aq(O_{2^n}) \end{pmatrix} = \begin{pmatrix} a-\delta+\beta \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{1}{2^{n-1}}M\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2^n} \end{pmatrix}. \tag{11}$$

We have taken $q(O_1)$, i.e. $q(x)$, to be zero. For $i = 2, 3, \ldots, 2^n$, (11) gives

$$\begin{aligned}
aq(O_i) &= -\frac{1}{2^{n-1}}\sum_{j=1}^{2^n} m_{ij}b_j \\
&= -\frac{1}{2^{n-1}}\sum_{j=1}^{2^n} m_{ij}(c(\eta_j)+\beta) \\
&= -\frac{1}{2^{n-1}}\sum_{j=1}^{2^n} m_{ij}c(\eta_j), \qquad (12)
\end{aligned}$$

where the last step follows from Lemma 1.

Step 5: Eliminating a. In view of (10) and (12), (6) becomes

$$a - |\delta - \beta| - \frac{1}{2^{n-1}} \sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} m_{ij} c(\eta_j)| \geq 0. \tag{13}$$

To turn this into a meaningful cancellativity criterion, we must eliminate a. This can be accomplished by comparing first components in the vector equation (11):

$$\begin{aligned} 0 &= a - \delta + \beta - \frac{1}{2^{n-1}} \sum_{j=1}^{2^n} m_{1,j} b_j \\ &= a - \delta + \beta - \frac{1}{2^{n-1}} \sum_{j=1}^{2^n} b_j \\ &= a - \delta + \beta - \frac{1}{2^{n-1}} \sum_{j=1}^{2^n} [c(\eta_j) + \beta] \\ &= a - \delta - \beta - \frac{1}{2^{n-1}} \sum_{j=1}^{2^n} c(\eta_j). \end{aligned}$$

Hence we may substitute $\delta + \beta + \frac{1}{2^{n-1}} \sum_{j=1}^{2^n} c(\eta_j)$ for a in (13) to obtain

$$\delta + \beta + \frac{1}{2^{n-1}} \sum_{j=1}^{2^n} c(\eta_j) - |\delta - \beta| - \frac{1}{2^{n-1}} \sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} c(\eta_j) m_{ij}| \geq 0,$$

$$\text{or } \delta + \beta - |\delta - \beta| \geq \frac{1}{2^{n-1}} [\sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} c(\eta_j) m_{ij}| - \sum_{j=1}^{2^n} c(\eta_j)].$$

If $\beta \geq \delta$ then the expression on the left equals 2δ; if $\beta \leq \delta$ then it equals 2β. Thus we have

$$\begin{aligned} 2 \min\{\beta, \delta\} &\geq \frac{1}{2^{n-1}} [\sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} c(\eta_j) m_{ij}| - \sum_{j=1}^{2^n} c(\eta_j)] \\ \text{or } \min\{\beta, \delta\} &\geq \frac{1}{2^n} [\sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} c(\eta_j) m_{ij}| - \sum_{j=1}^{2^n} c(\eta_j)]. \end{aligned}$$

Hence part (a) of the theorem holds with

$$T_x(c) = \frac{1}{2^n}[\sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} c(\eta_j)m_{ij}| - \sum_{j=1}^{2^n} c(\eta_j)]. \tag{14}$$

Proof of (b). By a theorem of Holley and Stroock (1979, equation 2.17; see also Theorem III.5.1 in Liggett (1985)), the system is ergodic if

$$\inf_{x\in S}\ a_x[1 - \sum_{F\subset\mathcal{N}_x} |q(x,F)|] > 0.$$

The construction of $T_x(c)$ in part (a) shows that $\min\{\beta_x, \delta_x\} - T_x(c)$ is just

$$\frac{1}{2}a_x[1 - \sum_{F\subset\mathcal{N}_x} |q(x,F)|].$$

Hence, if the infimum of the former is positive, ergodicity follows.

Corollary 1 *If $c(\cdot,\cdot)$ is the flip-rate function of a (non-noisy) voter model then*

$$sup_{x\in S}[\sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} c(\eta_j)m_{ij}| - \sum_{j=1}^{2^n} c(\eta_j)] \geq 0.$$

Proof. The expression on the left is simply $\sup_{x\in S}2^nT_x(c)$. Suppose this is negative. Then

$$\begin{aligned} 0 &< -\sup_{x\in S}T_x(c) \\ &\leq \inf_{x\in S}[\min\{\beta_x,\delta_x\} - T_x(c)], \end{aligned}$$

for any choices of β_x, δ_x. By part (b) of the theorem, the system is then ergodic for any choices of β_x, δ_x. Taking $\beta_x = \delta_x = 0$ for each $x \in S$, we conclude that the (non-noisy) voter model with flip-rate function c is ergodic. But this is absurd since by definition a voter model has at least two invariant distributions. The corollary follows.

Remarks.

1. If we are given only the noisy flip rates $c'(x,\eta)$ and wish to know if the resulting system is cancellative, the following alternative formula may be used:

$$T'_x(c') = \beta_x + \frac{1}{2^n}[\sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} c'(x,\eta_j)m_{ij}| - \sum_{j=1}^{2^n} c'(x,\eta_j)]. \tag{15}$$

   We have $T'_x(c') = T_x(c)$ for $c(x,\cdot)$ the underlying noiseless flip-rate function; this may be seen by recalling the definition of c' and using the fact that, by Lemma 1, $\sum_{j=1}^{2^n} m_{ij} = 0$ for $i = 2,3,\ldots,2^n$. Hence the Theorem may be applied directly even when only the noisy rates are given, e.g., for the Ising model (see Section 4).

2. It appears that similar techniques could reveal which contact processes (see Liggett (1997), §3) are cancellative.

3 Application: Cox-Durrett voter models

Formulas (14) and (15) for the threshold $T_x(c)$ or $T'_x(c')$ may be impractical for many specific cases. In this section we express the threshold in a more useful form for a very important subset of the NVM's.

Cox and Durrett (1991) considered translation-invariant voter models for which the flip rates depend solely on the number of disagreeing neighbours. More formally, for each (x,η), $c(x,\eta) = f(k)$ where $k = \#\{y \in \mathcal{N}_x : \eta(y) \neq \eta(x)\}$, with $f(0) = 0$. We now employ combinatorial arguments to derive a special expression for $T_x(c)$ in the Cox-Durrett case. Since these models are translation-invariant, the subscript in $T_x(c)$ will be omitted.

As in the proof of the Theorem, let $\eta_1,\ldots,\eta_{2^n}$ have distinct restrictions to $\mathcal{N}_x$ with $\eta_i(x) = 0$ for each i. Recalling (14) and letting $k_j = \#\{y \in \mathcal{N}_x : \eta_j(y) = 1\}$, we have

$$T(c) = \frac{1}{2^n}[\sum_{i=2}^{2^n} |\sum_{j=1}^{2^n} f(k_j)m_{ij}| - \sum_{t=1}^{2^n} f(k_t)]. \tag{16}$$

We rewrite this by first noting that

$$\sum_{j=1}^{2^n} f(k_j)m_{ij} \quad = \quad \sum_{s=0}^{n} \sum_{j:k_j=s} f(k_j)m_{ij}$$

$$= \sum_{s=0}^{n} f(s)[\#\{j : k_j = s, m_{ij} = 1\} - \#\{j : k_j = s, m_{ij} = -1\}] \tag{17}$$

Since $\#\{j : k_j = s, m_{ij} = 1\}$ and $\#\{j : k_j = s, m_{ij} = -1\}$ add up to $\binom{n}{s}$, (17) implies

$$\sum_{j=1}^{2^n} f(k_j) m_{ij} = \sum_{s=0}^{n} f(s)[2\#\{j : k_j = s, m_{ij} = 1\} - \tbinom{n}{s}], \tag{18}$$

Next observe that

$$\begin{aligned} \#\{j : k_j = s, m_{ij} = 1\} &= \#\{j : \ s \text{ neighbours are occupied in } \eta_j, \\ &\qquad \text{and an even number of these are among} \\ &\qquad \text{the } k_i \text{ occupied sites of } \eta_i\} \\ &= \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \tbinom{k_i}{2l}\tbinom{n-k_i}{s-2l}. \end{aligned} \tag{19}$$

By (18) and (19),

$$\sum_{j=1}^{2^n} f(k_j) m_{ij} = \sum_{s=0}^{n} f(s)[2 \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \tbinom{k_i}{2l}\tbinom{n-k_i}{s-2l} - \tbinom{n}{s}]. \tag{20}$$

Note also that

$$\sum_{t=1}^{2^n} f(k_t) = \sum_{u=0}^{n} \tbinom{n}{u} f(u). \tag{21}$$

Equations (20) and (21) transform (16) into

$$T(c) = \frac{1}{2^n}\left(\sum_{i=2}^{2^n} \left| \sum_{s=0}^{n} f(s)[2 \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \tbinom{k_i}{2l}\tbinom{n-k_i}{s-2l} - \tbinom{n}{s}] \right| - \sum_{u=0}^{n} \tbinom{n}{u} f(u) \right). \tag{22}$$

Since $\sum_{s=0}^{n} f(s)[2 \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \binom{k_i}{2l}\binom{n-k_i}{s-2l} - \binom{n}{s}]$ depends on i only through k_i, and since $k_i = r$ $(r = 1, 2, \ldots, n)$ for exactly $\binom{n}{r}$ values of i, (22) becomes

$$T(c) = \frac{1}{2^n}\left(\sum_{r=1}^{n} \tbinom{n}{r} \left| \sum_{s=0}^{n} f(s)[2 \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \tbinom{r}{2l}\tbinom{n-r}{s-2l} - \tbinom{n}{s}] \right| - \sum_{u=0}^{n} \tbinom{n}{u} f(u) \right). \tag{23}$$

Define an NVM corresponding to the given Cox-Durrett voter model by letting

$$f'(k) = f(k) + f'(0),\ k = 0, 1, \ldots, n, \tag{24}$$

for some $f'(0) \geq 0$ which represents both the spontaneous birth and the spontaneous death rate. As argued in Remark 1 at the end of the previous section, the threshold ((23) in this case) can alternatively be expressed in terms of the noisy flip-rate function (see (15)):

$$T'(c') = f'(0) + \frac{1}{2^n}\left(\sum_{r=1}^{n} \binom{n}{r} \left| \sum_{s=0}^{n} f'(s)[2 \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \binom{r}{2l}\binom{n-r}{s-2l} - \binom{n}{s}] \right| - \sum_{u=0}^{n} \binom{n}{u} f'(u)\right). \tag{25}$$

This leads immediately to

Corollary 2 *A noisy voter model with flip-rate function f', derived by (24) from a Cox-Durrett voter model with flip-rate function f, is cancellative if and only if $D_{f'} \geq 0$, where*

$$D_{f'} = \sum_{u=0}^{n} \binom{n}{u} f'(u) - \sum_{r=1}^{n} \binom{n}{r} \left| \sum_{s=0}^{n} f'(s)[2 \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} \binom{r}{2l}\binom{n-r}{s-2l} - \binom{n}{s}] \right|. \tag{26}$$

Proof. By (25), $D_{f'} \geq 0$ is equivalent to $f'(0) \geq T'(c')$, which is equivalent to cancellativity by the Theorem since $\beta_x = \delta_x = f'(0)$ for all x.

4 Sub-application: Ising models

For $n = 4$ we have

$$\begin{aligned} D_{f'} \ = \ & \sum_{u=0}^{4} \binom{4}{u} f'(u) - 4|f'(0) + 2f'(1) - 2f'(3) - f'(4)| \\ & +4|f'(0) - 2f'(1) + 2f'(3) - f'(4)| + 6|f'(0) - 2f'(2) + f'(4)| \\ & +|f'(0) - 4f'(1) + 6f'(2) - 4f'(3) + f'(4)|. \end{aligned} \tag{27}$$

In particular, for the two-dimensional Ising model, whose flip rates are given by

$$f'(k) = e^{(2k-4)\beta} \text{ for } k = 0, 1, 2, 3, 4, \tag{28}$$

each of the expressions in (27) within absolute value signs can be shown to have the same sign for all $\beta \geq 0$. Letting $D(\beta)$ be $D_{f'}$ for f' as given in (28), we can then compute

$$D(\beta) = 2e^{-4\beta} + 8e^{-2\beta} + 12 + 8e^{2\beta} - 14e^{4\beta}.$$

The cancellativity inequality $D(\beta) \geq 0$ can then be shown to hold iff

$$\beta \leq -\frac{1}{2} ln(\sqrt[4]{8} - 1) \approx .1915.$$

Thus the two-dimensional Ising model is cancellative for parameter values up to .1915. By contrast, the model is ergodic for values all the way up to $\beta_c = \frac{1}{2} ln(1 + \sqrt{2}) \approx .441$.

In $d > 2$ dimensions, the flip rates are

$$f'(k) = e^{2\beta(k-d)} \quad (k = 0, 1, \ldots, 2d)$$

and $D(\beta) = D_{f'}$ becomes even more unwieldy than (27); but Mathematica has no problem graphing the function D against β regardless of dimension. The graphs for two, three and four dimensions are given in Figures 1, 2 and 3, respectively. The x-intercept in each graph is evidently the supremum of the β values rendering the Ising model cancellative for that dimensionality. This supremum provides a lower bound for β_c, the critical value such that the d-dimensional Ising model is ergodic for $\beta < \beta_c$ and nonergodic for $\beta > \beta_c$ (see Holley (1974) and Liggett (1985), Theorem IV.1.31). Through computer calculations of $D(\beta)$, we thus obtain $\beta_c \geq .1230$ for $d = 3$ and $\beta_c \geq .0907$ for $d = 4$. Of course, the comparison in the previous paragraph for $d = 2$ casts considerable doubt on the sharpness of these bounds.

The one-dimensional Ising model is well known to be ergodic for all $\beta \geq 0$. It is cancellative, however, only for $\beta \leq \frac{1}{2}\ln(1+\sqrt{2})$—i.e., exactly those values of β for which there is no phase transition in the two-dimensional case. To see this, take $n = 2$ and $f'(k) = e^{2\beta(k-1)}$ for $k = 0, 1, 2$ (the one-dimensional Ising model flip-rate function) in (26) to obtain

$$D(\beta) = 2(2 + e^{-2\beta} - e^{2\beta}),$$

whence $D(\beta) \geq 0$ iff $\beta \leq \frac{1}{2}\ln(1 + \sqrt{2})$.

5 Proofs of the lemmata concerning M

In this section we prove Lemmata 1 and 2, which are restated for convenience.

Lemma 1 $\frac{1}{2^n}M\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$.

Proof. $\frac{1}{2^n}M\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ has first component

$$\frac{1}{2^n}\sum_{j=1}^{2^n} m_{1j}1=\frac{1}{2^n}\sum_{j=1}^{2^n}1=1.$$

For $i=2,3,\ldots,2^n$, let z_i be an occupied site of η_i. Then $\frac{1}{2^n}M\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ has i^{th} component

$$\begin{aligned}\frac{1}{2^n}\sum_{j=1}^{2^n}\chi_{E_j}(\eta_i)\cdot 1 &= \frac{1}{2^n}[\sum_{j:z_i\notin E_j}\chi_{E_j}(\eta_i)+\sum_{z_i\in E_j}\chi_{E_j}(\eta_i)]\\ &= \frac{1}{2^n}\sum_{j:z_i\notin E_j}[\chi_{E_j}(\eta_i)+\chi_{E_j\cup\{z_i\}}(\eta_i)]\\ &= 0,\end{aligned}$$

since $\chi_{E_j}(\eta_i)$ and $\chi_{E_j\cup\{z_i\}}(\eta_i)$ are of opposite sign. This establishes Lemma 1.

Lemma 2 $\frac{1}{2^n}M^2=I$.

Proof. $\frac{1}{2^n}M^2$ has (i,j) entry

$$\begin{aligned}\frac{1}{2^n}\sum_{l=1}^{2^n} m_{il}m_{lj} &= \frac{1}{2^n}\sum_{l=1}^{2^n}\chi_{E_l}(\eta_i)\chi_{E_j}(\eta_l)\\ &= \sum_{l=1}^{2^n}\frac{1}{2^n}\chi_{E_i}(\eta_l)\chi_{E_j}(\eta_l)\\ &= \frac{1}{2^n}\sum_{l=1}^{2^n}\chi_{E_i\triangle E_j}(\eta_l).\end{aligned}$$

If $i=j$ then each summand in the last expression is $\chi_\emptyset(\eta_l)=1$; thus the diagonal entries of $\frac{1}{2^n}M^2$ are all 1 as required. If $i\neq j$ then $E_i\triangle E_j$ contains some element z_{ij}; without loss of generality the η_l's are numbered so that, for $l=1,2,\ldots,2^{n-1}$, η_l and $\eta_{l+2^{n-1}}$ differ only at z_{ij}, and so the (i,j) entry of $\frac{1}{2^n}M^2$ is

$$\frac{1}{2^n}\sum_{l=1}^{2^{n-1}}[\chi_{E_i\triangle E_j}(\eta_l)+\chi_{E_i\triangle E_j}(\eta_{l+2^{n-1}})]$$

$=0$, since $\chi_{E_i\triangle E_j}(\eta_l)=\chi_{E_i\triangle E_j}(\eta_{l+2^{n-1}})$ for $l=1,2,\ldots,2^{n-1}$. We conclude that $\frac{1}{2^n}M^2=I$.

Remark. A $k\times k$ matrix A such that $A^tA=AA^t=kI$ is called a Hadamard matrix (Hall (1967), Chapter 14). M's symmetry and Lemma 2 imply that M is a Hadamard matrix.

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REFERENCES

1. V. Belitsky and B.L. Granovsky, On the dynamics of the stochastic model of adsorption-desorption on a finite lattice, Commun. Statist.— Stochastic Models 7 (1991) 327-341.

2. P. Clifford and A. Sudbury, A model for spatial conflict, Biometrika 60 (1973) 581-588.

3. J.T. Cox and R. Durrett, Nonlinear voter models, in R. Durrett and H. Kesten, eds., Random Walks, Brownian Motion, and Interacting Particle Systems: A Festschrift in Honor of Frank Spitzer (Birkhäuser, Boston, 1991), pp. 189-201.

4. R. Durrett, An introduction to infinite particle systems, Stochastic Process. Appl. 11 (1981) 109-150.

5. D. Griffeath, Additive and Cancellative Interacting Particle Systems (Springer Lecture Notes in Mathematics, Vol. 724, 1979).

6. B.L. Granovsky and N. Madras, The noisy voter model, Stochastic. Process. Appl. 55 (1995) 23-43.

7. M. Hall, Combinatorial Theory (Blaisdell, Waltham, MA, 1967).

8. R. Holley, Recent results on the stochastic Ising model, Rocky Mountain J. Math. 4 (1974) 479-496.

9. R. Holley and T.M. Liggett, Ergodic theorems for weakly interacting systems and the voter model, Ann. Probab. 3 (1975) 643-663.

10. R. Holley and D. Stroock, Dual processes and their application to infinite interacting systems, Adv. Math. 32 (1979) 149-174.

11. T.M. Liggett, Interacting Particle Systems (Springer, New York, 1985).

12. T.M. Liggett, Stochastic models of interacting systems, Ann. Probab. 25 (1997) 1-29.

13. A. Sudbury and P. Lloyd, Quantum operators in classical probability theory: II. The concept of duality in interacting particle systems, Ann. Probab. 23 (1995) 1816-1830.