STA447/2006 (Stochastic Processes), Winter 2008

Homework #3

Due: In class by 6:10 p.m. **sharp** on Thursday March 20. (If you prefer, you may bring your assignment to the instructor's office, Sidney Smith Hall room 6024, any time before it is due; slide it under the door if he is not in.) **Warning: Late homeworks, even by one minute, will be penalised!** (See the "Grade-Related Course Policies".)

Reminder: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Copying other solutions is strictly prohibited!

THE ASSIGNMENT: [Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly.]

Include at the top of the first page: Your name and student number, and whether you are enrolled in STA447 or STA2006.

1. Consider the Markov chain (from the midterm) with state space $S = \{1, 2, 3\}$, and transition probabilities given by $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, with $p_{ij} = 0$ otherwise.

(a) [5 points] Compute a precise formula for $p_{12}^{(n)}$, for any positive integer n.

(b) [3 points] Compute $\sum_{n=1}^{N} p_{12}^{(n)}$ for any positive integer N.

(c) [2 points] Using your formula from part (b), prove that $\sum_{n=1}^{\infty} p_{12}^{(n)} = \infty$.

2. Let $\{Y_n\}$ be a supermartingale, and let T be a stopping time with $\mathbf{P}(T < \infty) = 1$. Assume that $Y_{\min(n,T)} \geq 0$, i.e. that $\{Y_n\}$ is non-negative up to time T.

(a) [5 points] Let $M > 0$, and let $W_n = \min(Y_n, M)$. Prove that $\{W_n\}$ is also a supermartingale. [Hint: consider separately the cases $W_n = M$ and $W_n < M$, and remember that $W_{n+1} \leq Y_{n+1}$.]

(b) [2 points] Prove that $\mathbf{E}(W_T) \leq \mathbf{E}(W_0) \leq \mathbf{E}(Y_0)$. [Note: you may use without proof the obvious generalisation of our martingale stopping theorem corollary, i.e. that if $\{X_n\}$ is a supermartingale with stopping time T with $\mathbf{P}(T < \infty) = 1$, and $\{X_n\}$ is bounded up to time T, then $\mathbf{E}(X_T) \leq \mathbf{E}(X_0)$.]

(c) [3 points] Prove that $\mathbf{E}(Y_T) \leq \mathbf{E}(Y_0)$. [Hint: Take (with justification) the limit $M \to \infty$.]

3. Let $\{X_n\}$ be a Markov chain, with $X_0 = a > 0$, and that there is $\beta > 0$ with $\mathbf{E}(X_{n+1} - X_n \,|\, X_n = x) \leq -\beta$ for all $x > 0$. Let $T = \min\{n \geq 1; X_n = 0\}$, and assume that $T = \min\{n \geq 1; X_n \leq 0\}$ (i.e., that $\{X_n\}$ always hits 0 before becoming negative).

(a) [5 points] Let $Y_n = X_n + \beta\, n$. Prove that $\{Y_{\min(n,T)}\}$ is a supermartingale.

(b) [2 points] Prove that $Y_{\min(n,T)} \geq 0$.

(b) [3 points] Prove that $\mathbf{E}(T) \leq a/\beta$. [Hint: don't forget the previous question.]

4. ("tortoise and hare") Let $0 < \epsilon < \beta$, and let $U = \beta + \epsilon$ or $\beta - \epsilon$ with probability 1/2 each. Let $\{Z_n\}_{n=1}^{\infty}$ be conditionally i.i.d. given U, with $\mathbf{P}(Z_n = -1 \,|\, U) = U$ and $\mathbf{P}(Z_n = 0 \,|\, U) = 1 - U$. Let a be a positive integer, and let $X_n = a + Z_1 + \ldots + Z_n$, and let $T = \min\{n \geq 1 : X_n = 0\}$.

(a) [2 points] Prove that $X_{\min(n,T)} \geq 0$.

(b) [3 points] Prove that $\mathbf{E}(X_{n+1} - X_n \,|\, X_n = x) = -\beta$ for all $x > 0$.

(c) [6 points] Prove that $\mathbf{E}(T) = \frac{1}{2}\frac{a}{\beta+\epsilon} + \frac{1}{2}\frac{a}{\beta-\epsilon}$. [Hint: Recall that $\mathbf{E}(T) = \sum_z \mathbf{P}(U = z)\, \mathbf{E}(T \,|\, U = z)$, and don't forget the mean of a geometric random variable.]

(d) [2 points] Prove that $\mathbf{E}(T) > a/\beta$.

(e) [2 points] Why does this fact not contradict the results of the previous question?

5. Text exercise 7.20 (p. 154), parts (a) [5 points] and (b) [5 points].

6. [5 points] Text exercise 7.27 (p. 155).

7. [10 points] Text exercise 7.10 (p. 153). [Note: the question is asking for the <u>mean</u> time. Hint: One way to proceed is to first compute $\mathbf{P}(T < t)$, and then use the general formula $\mathbf{E}(T) = \int_0^\infty \mathbf{P}(T \geq t)\, dt$. Alternatively, find the mean time until the first person catches a fish, then the mean time after that until the second person catches a fish, etc.]