STA447/2006 HW#4 Solutions – OUTLINE ONLY

NOTE: What follows is just an outline of the homework solutions, taken from the textbook's solutions manual. These solutions may be incorrect or incomplete; more complete explanations may be required to earn full points on the homework.

8.17. X_t is a birth and death chain. The death rates (jumping into the lake) are $\mu_i = i$, while the birth rates (jumping out of the lake) are $\lambda_i = 2(3-i)$. Setting $\pi(0) = c$ and plugging into the recursion (3.5) gives

$$\pi(1) = \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{6}{1} \cdot c = 6c$$
$$\pi(2) = \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{4}{2} \cdot 6c = 12c$$
$$\pi(3) = \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{2}{3} \cdot 12c = 8c$$

Adding up the π's gives $(8 + 12 + 6 + 1) = 27c$ so $c = 1/27$ and we have

$$\pi(3) = \frac{8}{27} \quad \pi(2) = \frac{12}{27} \quad \pi(1) = \frac{6}{27} \quad \pi(0) = \frac{1}{27}$$

(b) Each frog is a two state Markov chain that stays in the sun 2/3's of the time and in the lake 1/3 of the time. Thus the number in the sun should be Binomial(3,2/3). Since the Binomial probabilities are

$$\pi(3) = (2/3)^3 \quad \pi(2) = 3(2/3)^2(1/3) \quad \pi(1) = 3(1/3)^2(2/3) \quad \pi(0) = (1/3)^3$$

this agrees with the previous answer.

8.19. Let X_t be the number of working machines. X_t is a birth and death chain. Taking into account the number of repairmen working $\lambda_2 = 1/2$, $\lambda_1 = \lambda_0 = 1$. The death rate is proportional to the number of machines working so $\mu_1 = 1/20$, $\mu_2 = 2/10$ and $\mu_3 = 3/20$. Setting $\pi(0) = c$ and plugging into the recursion (3.5) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{1}{1/20} \cdot c = 20c \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{1}{2/20} \cdot 20c = 200c \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{1/2}{3/20} \cdot 200c = 2000c/3
\end{aligned}$$

Adding up the π's gives $(2000 + 600 + 60 + 3)c/3 = 2663c/3$ so $c = 3/2663$ and we have

$$\pi(3) = \frac{2000}{2663} \quad \pi(2) = \frac{600}{2663} \quad \pi(1) = \frac{60}{2663} \quad \pi(0) = \frac{3}{2663}$$

(b) $\pi(0) + \pi(1) = 63/2663 = .0237$ of the time. (c) $(6000 + 1200 + 60)/2663 = 7260/2663 = 2.726$.

8.25. Let X_t be the number of customers in the system. X_t is a birth and death chain with $\lambda_n = \lambda$ for all $n \geq 0$, and $\mu_n = \mu + (n-1)\delta$. It follows from (3.6) that

$$\pi(n+1) = \frac{\lambda_n}{\mu_{n+1}} \cdot \pi(n)$$

If $\delta > 0$, we have $\lambda_n/\mu_{n+1} \to 0$ as $n \to \infty$. Hence if N is large enough and $n \geq N$ then $\lambda_n/\mu_{n+1} \leq 1/2$ and the desired conclusion follows from the argument in Example 3.5. (b) When $\delta = \mu$, $\mu_n = n\mu$ and

$$\frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n} = \frac{\lambda^n}{\mu^n} \cdot \frac{1}{1 \cdot 2 \cdots n} = \frac{(\lambda/\mu)^n}{n!}$$

It follows that the stationary distribution is Poisson with mean λ/μ.

5.4. Using (1.1) we have

$$P(M > n) = P(t_1 + \cdots + t_n \le 1)$$
$$= \int_{t_1=0}^{1} \int_{t_2=0}^{1-t_1} \cdots \int_{t_n=0}^{1-(t_1+\cdots+t_{n-1})} 1 \, dt_n \cdots dt_2 \, dt_1 = \frac{1}{n!}$$

From this and (1.3) it follows that

$$EM = \sum_{n=0}^{\infty} P(M > n) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

(b) Wald's equation, (1.5), implies that

$$ET_M = EM \cdot Et_i$$

Since t_i is uniform on $(0,1)$ we have $Et_i = 1/2$. Using (a) now

$$E(T_M - 1) = \left(e \cdot \frac{1}{2}\right) - 1 = .35914$$

6.1. Using the definition of conditional probability gives $P(B_s = x | B_t = z)$ is

$$\frac{\exp(-x^2/2s)}{(2\pi s)^{1/2}} \cdot \frac{\exp(-(z-x)^2/2(t-s))}{(2\pi(t-s))^{1/2}} \cdot \frac{(2\pi t)^{1/2}}{\exp(-z^2/2(t-s))}$$
$$= (2\pi)^{-1/2} \left(\frac{t}{s(t-s)}\right)^{1/2} \exp\left\{-\frac{x^2}{2s} - \frac{(x-z)^2}{2(t-s)} - \frac{z^2}{2(t-s)}\right\}$$

The expression in braces is

$$-\frac{1}{2}\left(\frac{t}{s(t-s)}x^2 - \frac{2}{(t-s)}xy + \frac{s}{t(t-s)}z^2\right) = -\frac{t}{2s(t-s)}\left(x - \frac{zs}{t}\right)^2$$

so the distribution is normal$(zs/t, s(t-s)/t)$.

6.4. (a) $EY_t = \int_0^t EB_s\, ds = 0$. (b) To dto this, we use a trick

$$\begin{aligned} EY_t^2 &= E\left(\int_0^t B_s\, ds\right)^2 = E\left(\int_0^t B_r\, dr\right)\left(\int_0^t B_s\, ds\right) \\ &= E\left(\int_0^t \int_0^t B_r B_s\, dr\, ds\right) = 2\int_0^t \int_0^s EB_r B_s\, dr\, ds \\ &= 2\int_0^t \int_0^s r\, dr\, ds = \int_0^t s^2\, ds = t^3/3 \end{aligned}$$

(c) Clearly $EY_sY_t = EY_s^2 + EY_s(Y_t - Y_s)$, so we only have to compute the seond term. To do this we imitate the computation in (b)

$$\begin{aligned} EY_s(Y_t - Y_s) &= E\left(\int_0^s B_r\, dr \cdot \int_s^t B_u\, du\right) \\ &= \int_0^s \int_s^t EB_r B_u\, du\, dr \\ &= \int_0^s \int_s^t dur\, dr\, ds = (t-s)\int_0^s r\, dr = (t-s)s^2/2 \end{aligned}$$

6.39. By (4.3) we want to find a probability distribution so that the two stocks are each martingales, i.e.,

$$20p_1 + 10p_2 - 16p_3 = 0 \qquad -20p_1 + 5p_2 + 10p_3 = 0$$

Substituting $p_3 = 1 - p_1 - p_2$ we have

$$36p_1 + 26p_2 = 16 \qquad -30p_1 - 5p_2 = -10$$

Multiplying the second equation by $6/5$ and adding it to the first we have $20p_2 = 4$ so $p_2 = .2$. Solving now gives $p_1 = .3$ and $p_3 = .5$. An option to buy Netscape at 50 pays off 0 in case 1, 5 in case 2, and 10 in case 3, so the option is worth $0p_1 + 5p_2 + 10p_3 = 0 + 1 + 5 = 6$.