STA 2111 (Graduate Probability I), Fall 2023

Homework #1 Assignment: worth 10% of final course grade.

Due: in class by 10:10 a.m. **sharp** (Toronto time) on Thursday Oct 5.

GENERAL NOTES:

- Homework assignments are to be solved by each student individually. You may discuss questions in general terms with other students, and look up general topics in books and internet, but you must solve the problems on your own, and do all of your own writing.

- You should provide very complete solutions, including explaining all of your reasoning very clearly. Please submit your assignment as hard copy at the beginning of class.

- Please also include your name and student number and department and program and year and e-mail address at the beginning of your assignment – thank you.

- **Neatness bonus:** If your homework is neat and easy to read, e.g. typeset in tex/latex or otherwise typed or printed very clearly, then you might receive up to 5% bonus points.

- **Late penalty:** 1–5 minutes late is −5%; 5–15 minutes late is −10%; otherwise if x days late then $-20\% \times \text{ceiling}(x)$. So, don't be late!

THE ACTUAL ASSIGNMENT:

1. [**4**] Suppose that $\Omega = \{1,2\}$, and $\mathcal{F} = 2^\Omega$ is the collection of all subsets of Ω, and $\mathbf{P} : \mathcal{F} \to [0,1]$ with $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}(\Omega) = 1$. Prove that $\mathbf{P}$ is countably additive if and only if $\mathbf{P}\{1\} \geq 0$, $\mathbf{P}\{2\} \geq 0$, and $\mathbf{P}\{1\} + \mathbf{P}\{2\} = 1$.

2. Let $\Omega = \{1,2,3,4\}$. Determine whether or not each of the following is a σ-algebra.

(a) [**4**] $\mathcal{F}_1 = \Big\{\emptyset,\ \{1,3\},\ \{2,4\},\ \{1,2,3,4\}\Big\}$.

(b) [**4**] $\mathcal{F}_2 = \Big\{\emptyset,\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\},\ \{1,2,3,4\}\Big\}$.

3. Let $\Omega = \{1,2,3,4\}$, and let $\mathcal{J} = \Big\{\emptyset, \{1\}, \{2\}, \{3,4\}, \Omega\Big\}$. Define $\mathbf{P} : \mathcal{J} \to [0,1]$ by $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}\{1\} = 1/8$, $\mathbf{P}\{2\} = 1/4$, $\mathbf{P}\{3,4\} = 5/8$, and $\mathbf{P}(\Omega) = 1$.

(a) [**3**] Prove that $\mathcal{J}$ is a semi-algebra.

(b) [**5**] Compute $\mathbf{P}^*(A)$ and $\mathbf{P}^*(A^C)$, where $A = \{2,3\} \subseteq \Omega$ and $\mathbf{P}^*$ is outer measure.

(c) [**5**] Determine whether or not $A \in \mathcal{M}$, where $\mathcal{M}$ is the σ-algebra constructed in the proof of the Extension Theorem. [Hint: Perhaps consider the case $E = \Omega$.]

4. [**5**] Suppose that $\Omega = \mathbf{Z}$ is the set of all integers, and $\mathbf{P}$ is defined for all $A \subseteq \Omega$ by $\mathbf{P}(A) = 0$ if A is finite, and $\mathbf{P}(A) = 1$ if A is infinite. Is $\mathbf{P}$ finitely additive?

5. Let $\Omega = \mathbf{Z}$ be the set of all integers, and let

$$\mathcal{B} \;=\; \{A \subseteq \Omega : \text{either } A \text{ is finite or } A^C \text{ is finite}\}\,.$$

Let $\mathbf{P} : \mathcal{B} \to [0,1]$ by $\mathbf{P}(A) = 0$ if A is finite, and $\mathbf{P}(A) = 1$ if A^C is finite.

(a) [5] Is $\mathcal{B}$ an algebra (meaning that $\emptyset, \Omega \in \mathcal{B}$, and $\mathcal{B}$ is closed under complement and under finite union)?

(b) [5] Is $\mathcal{B}$ a σ-algebra?

(c) [5] Is $\mathbf{P}$ finitely additive on $\mathcal{B}$?

(d) [5] Is $\mathbf{P}$ countably additive on $\mathcal{B}$ (meaning that if $A_1, A_2, \ldots \in \mathcal{B}$, and if also $\bigcup_n A_n \in \mathcal{B}$, then $\mathbf{P}(\bigcup_n A_n) = \sum_n \mathbf{P}(A_n)$)?

6. [5] Prove that the extension $(\Omega, \mathcal{M}, \mathbf{P}^*)$ constructed in the proof of the Extension Theorem must be "complete", meaning that if $A \in \mathcal{M}$ with $\mathbf{P}^*(A) = 0$, and if $B \subseteq A$, then $B \in \mathcal{M}$. (It then follows from monotonicity that $\mathbf{P}^*(B) = 0$.)

7. For any interval $I \subseteq [0,1]$, let $\mathbf{P}(I)$ be the <u>length</u> of I.

(a) [5] Prove that if $I_1, I_2, \ldots, I_n$ is a <u>finite</u> collection of intervals, and if $\bigcup_{j=1}^n I_j \supseteq I_*$ for some interval I_*, then $\sum_{j=1}^n \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: Suppose I_j has left endpoint a_j and right endpoint b_j, and first re-order the intervals so $a_1 \leq a_2 \leq \ldots \leq a_n$.]

(b) [5] Prove that if $I_1, I_2, \ldots$ is a countable collection of <u>open</u> intervals, and if $\bigcup_{j=1}^\infty I_j \supseteq I_*$ for some <u>closed</u> interval I_*, then $\sum_{j=1}^\infty \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: You may use the <u>Heine-Borel Theorem</u>, which says that if a collection of open intervals contain a closed interval, then some <u>finite sub-collection</u> of the open intervals also contains the closed interval.]

(c) [5] Prove that if $I_1, I_2, \ldots$ is *any* countable collection of intervals, and if $\bigcup_{j=1}^\infty I_j \supseteq I_*$ for *any* interval I_*, then $\sum_{j=1}^\infty \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. (Note: This is the "countable monotonicity" property needed to apply the Extension Theorem for the Uniform[0,1] distribution, to guarantee that $\mathbf{P}^*(I) \geq \mathbf{P}(I)$.) [Hint: Extend the interval I_j by $\epsilon\, 2^{-j}$ at each end, and decrease I_* by ϵ at each end, while making I_j open and I_* closed. Then use part (b).]

(d) [5] Suppose we instead defined $\mathbf{P}(I)$ to be the <u>square</u> of the length of I. Show that in that case, the conclusion of part (c) would <u>not</u> hold.

[END; total points = 75]