STA 2111 (Graduate Probability I), Fall 2023

Homework #2 Assignment: worth 10% of final course grade.

Due: in class by 10:10 a.m. sharp (Toronto time) on Thursday Nov. 23.

GENERAL NOTES:

- Homework assignments are to be solved by each student individually. You may discuss questions in general terms with other students, but you must solve them on your own, including doing all of your own computing and writing.

- You should provide very complete solutions, including explaining all of your reasoning clearly. Please submit your assignment as hard copy at the beginning of in class.

- **Late penalty:** 1–5 minutes late is −5%; 5–15 minutes late is −10%; otherwise if x days late then $-20\% \times \text{ceiling}(x)$. So, don't be late!

THE ACTUAL ASSIGNMENT:

1. **[5]** Let $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ be Lebesgue measure on $[0,1]$. Consider a second probability triple, $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, defined as follows: $\Omega_2 = \{1,2\}$, $\mathcal{F}_2$ consists of all subsets of Ω_2, and $\mathbf{P}_2 : \mathcal{F}_2 \to \mathbf{R}$ is defined by $\mathbf{P}_2(\emptyset) = 0$, $\mathbf{P}_2\{1\} = \frac{1}{4}$, $\mathbf{P}_2\{2\} = \frac{3}{4}$, $\mathbf{P}_2\{1,2\} = 1$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the product measure of $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$. Find (with explanation) two disjoint subsets $A, B \in \mathcal{F}$ such that $\mathbf{P}(A) = \mathbf{P}(B) = \frac{1}{5}$.

2. **[6]** Let $([0,1]^2, \mathcal{F}, \lambda)$ be Lebesgue measure on $[0,1]^2$, i.e. the product measure $\text{Unif}[0,1] \times \text{Unif}[0,1]$. Let A be the triangle $\{(x,y) \in [0,1]^2 \,;\, x+y < 1\}$. Prove $A \in \mathcal{F}$, and find $\lambda(A)$.

3. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability triple, let $B, C \in \mathcal{F}$ be two fixed events, and let

$$A_n = \begin{cases} B, & n \text{ odd} \\ C, & n \text{ even} \end{cases}$$

In terms of B and C:

(a) **[3]** Specify the events $\liminf_n A_n$ and $\limsup_n A_n$.

(b) **[3]** Specify the values $\liminf\limits_{n\to\infty} \mathbf{P}(A_n)$ and $\limsup\limits_{n\to\infty} \mathbf{P}(A_n)$.

(c) **[3]** Show directly why

$$\mathbf{P}\Big(\liminf_n A_n\Big) \;\le\; \liminf_{n\to\infty} \mathbf{P}(A_n) \;\le\; \limsup_{n\to\infty} \mathbf{P}(A_n) \;\le\; \mathbf{P}\Big(\limsup_n A_n\Big).$$

4. Consider infinite, independent, fair coin tossing, with $H_n = \{n^{\text{th}} \text{ coin is heads}\}$.

(a) **[4]** Compute $\mathbf{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lfloor 2\log_2 n \rfloor} \ i.o.)$.

(b) **[4]** Compute $\mathbf{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lfloor \log_2 n \rfloor} \ i.o.)$.

5. Let X be a non-negative random variable with $\mathbf{P}(X > 0) > 0$.

(a) [**4**] Prove that there exists some $n \in \mathbf{N}$ such that $\mathbf{P}(X \geq 1/n) > 0$.

(b) [**4**] Prove that $\mathbf{E}(X) > 0$.

6. Give examples of random variables X defined on Lebesgue measure on $[0, 1]$ with:

(a) [**3**] $\mathbf{E}(X^+) = \infty$ and $0 < \mathbf{E}(X^-) < \infty$.

(b) [**3**] $\mathbf{E}(X^-) = \infty$ and $0 < \mathbf{E}(X^+) < \infty$.

(c) [**3**] $\mathbf{E}(X^+) = \mathbf{E}(X^-) = \infty$.

(d) [**3**] $X \geq 0$ and $0 < \mathbf{E}(X) < \infty$ but $\mathbf{E}(X^2) = \infty$.

7. Let $\{X_n\}$ and X be random variables defined on Lebesgue measure on $[0, 1]$, with $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for each fixed $\omega \in [0, 1]$. Suppose for each $n \in \mathbf{N}$, $\mathbf{E}|X_n| < \infty$, and also $X_{n+1}(\omega) \leq X_n(\omega)$ for all $\omega \leq 1/2$, and $X_{n+1}(\omega) \geq X_n(\omega)$ for all $\omega > 1/2$.

(a) [**4**] Assuming $\mathbf{E}|X| < \infty$, prove that $\lim_{n\to\infty} \mathbf{E}(X_n) = \mathbf{E}(X)$.

(b) [**4**] Without assuming $\mathbf{E}|X| < \infty$, find an example where $\mathbf{E}(X)$ is undefined, and hence $\lim_{n\to\infty} \mathbf{E}(X_n) \neq \mathbf{E}(X)$.

8. For each of the following conditions, either give an example of a random variable X defined on Lebesgue measure on $[0, 1]$ satisfying those conditions, or prove that no such random variable exists.

(a) [**3**] $X \geq 0$, and $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \geq 7) = 1/2$.

(b) [**3**] $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \geq 7) = 1/2$.

(c) [**3**] $\mathbf{E}(X) = 3$, and $\mathbf{Var}(X) = 4$, and $\mathbf{P}(X \leq 0) = 1/2$.

9. Let $\{I_j\}_{j=1}^{\infty}$ be independent random variables, with $I_1 \sim \text{Uniform}\{0, 1, 2, \ldots, 9\}$, $I_2 \sim \text{Uniform}\{10, 11, 12, \ldots, 99\}$, $I_3 \sim \text{Uniform}\{100, 101, 102, \ldots, 999\}$, and in general, I_j is chosen uniformly from the set of all non-negative integers having j digits.
Then, define $\{X_n\}_{n=1}^{\infty}$ by: $X_n = 5$ if $n = I_j$ for some j, otherwise $X_n = 0$.

(a) [**5**] Prove or disprove that $\{X_n\} \to 0$ in probability.

(b) [**5**] Prove or disprove that $\{X_n\} \to 0$ almost surely.

[END; total points = 75]