STA 2111 (Graduate Probability I), Fall 2025

Homework #2 Assignment: worth 10% of final course grade.

Due: Thursday Nov. 13, either: **(a)** Hardcopy in class by 2:10 PM **sharp**, or **(b)** electronic submission (see instructions on course web page) by 1:45 PM **sharp**, both in Toronto time.

GENERAL NOTES:

- Homework assignments are to be solved by each student individually. You may discuss questions in general terms with other students, and look up general topics in books and internet. But you must solve the problems on your own, and do all of your own writing, without any assistance from other students nor from any AI/chatbot/chatGPT/etc software.

- You should provide very complete solutions, **EXPLAINING ALL REASONING** very clearly. Make your homework neat and readable, e.g. typeset in latex or printed clearly.

- **Late penalty:** 1–5 minutes late is -5%; 5–15 minutes late is -10%; otherwise if x days late then $-20\% \times \text{ceiling}(x)$. So, please don't be late!

THE ACTUAL ASSIGNMENT:

1. **[5]** Let $A_1, A_2, \ldots$ be independent events. Let Y be a random variable which is measurable with respect to $\sigma(A_n, A_{n+1}, \ldots)$ for each $n \in \mathbf{N}$. Prove there is some $x \in \mathbf{R}$ such that $\mathbf{P}(Y = x) = 1$. [Hints: You may wish to consider $\mathbf{P}(Y \le y)$. And don't forget Kolmogorov.]

2. Let X be a non-negative random variable with $\mathbf{P}(X > 0) > 0$.

(a) **[3]** Prove that there must exist some $n \in \mathbf{N}$ such that $\mathbf{P}(X \ge 1/n) > 0$.

(b) **[3]** Prove that we must have $\mathbf{E}(X) > 0$.

3. For each of the following conditions, either give an example of a random variable X defined on Lebesgue measure on $[0, 1]$ satisfying those conditions, or prove that no such random variable exists.

(a) **[3]** $X \ge 0$, and $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \ge 7) = 1/2$.

(b) **[3]** $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \ge 7) = 1/2$.

(c) **[3]** $\mathbf{E}(X) = 3$, and $\mathbf{Var}(X) = 4$, and $\mathbf{P}(X \le 0) = 1/2$.

4. Let Y be any non-negative random variable.

(a) **[3]** Prove that $Y\,\mathbf{1}_{Y>0} = Y$. [Hint: Consider separately $Y(\omega) = 0$ and $Y(\omega) > 0$.]

(b) **[5]** Assuming that $0 < \mathbf{E}(Y^2) < \infty$, prove that $\mathbf{P}(Y > 0) \ge [\mathbf{E}(Y)]^2 / \mathbf{E}(Y^2)$. [Hints: apply Cauchy-Schwarz with $X = \mathbf{1}_{Y>0}$. And don't forget part (a).]

5. Suppose X and $\{X_n\}$ are all random variables defined on some $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is countable and $\mathcal{F} = 2^{\Omega}$, and that $\{X_n\} \to X$ in probability.

(a) **[4]** Prove that if $\omega \in \Omega$ with $\mathbf{P}(\{\omega\}) > 0$, then for all $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $\mathbf{P}(|X_n - X| \ge \epsilon) < \mathbf{P}(\{\omega\})$ for all $n \ge N$.

(b) **[4]** Prove that $\{X_n\} \to X$ almost surely. [Hint: Part (a) might help.]

6. Let $\{I_j\}_{j=1}^{\infty}$ be independent random variables, with $I_1 \sim \text{Uniform}\{1, 2, \ldots, 9\}$, $I_2 \sim \text{Uniform}\{10, 11, 12, \ldots, 99\}$, $I_3 \sim \text{Uniform}\{100, 101, 102, \ldots, 999\}$, and in general, I_j is chosen uniformly from the set of all positive integers having j digits.
Then, define $\{X_n\}_{n=1}^{\infty}$ by: $X_n = 7$ if $n = I_j$ for some j, otherwise $X_n = 5$.

(a) [**4**] Prove or disprove that $\{X_n\} \to 5$ in probability.

(b) [**4**] Prove or disprove that $\{X_n\} \to 5$ almost surely.

7. Let $\{X_n\}$ be random variables defined on Uniform$[0, 1]$, with $|X_n(\omega)| < \infty$ and $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for each fixed $\omega \in [0, 1]$. Suppose for each $n \in \mathbf{N}$, $\mathbf{E}|X_n| < \infty$, and also $X_{n+1}(\omega) \le X_n(\omega)$ for all $\omega \le 1/2$, and $X_{n+1}(\omega) \ge X_n(\omega)$ for all $\omega > 1/2$.

(a) [**3**] Prove that $\lim_{n\to\infty} \mathbf{E}(X_n \mathbf{1}_{(\frac{1}{2},1]}) = \mathbf{E}(X \mathbf{1}_{(\frac{1}{2},1]})$.

(b) [**5**] Prove that $\lim_{n\to\infty} \mathbf{E}(X_n \mathbf{1}_{[0,\frac{1}{2}]}) = \mathbf{E}(X \mathbf{1}_{[0,\frac{1}{2}]})$. [Hint: What can you say about $Y_n = X_1 \mathbf{1}_{[0,\frac{1}{2}]} - X_n \mathbf{1}_{[0,\frac{1}{2}]}$? And, remember that $\mathbf{E}|X_1| < \infty$.]

(c) [**3**] Assuming $\mathbf{E}|X| < \infty$, prove that $\lim_{n\to\infty} \mathbf{E}(X_n) = \mathbf{E}(X)$.

(d) [**5**] Without assuming that $\mathbf{E}|X| < \infty$, find an example of such $\{X_n\}$ where $\mathbf{E}(X)$ is undefined, i.e. $\mathbf{E}(X^+) = \mathbf{E}(X^-) = \infty$, and hence $\lim_{n\to\infty} \mathbf{E}(X_n) \ne \mathbf{E}(X)$.

[END; total points = 60]