Time to L^2 for certain random walks on compact Lie groups*

(Notes in progress, 1994.)

Ursula Porod and Jeffrey S. Rosenthal

On the unitary group $U(N)$, consider the random walk with step distribution given by the pushforward of the measure $C_a(\sin(\theta/2))^a d\theta \times d\lambda$ under the map $(x, \theta) \mapsto x^{-1}\mathrm{diag}(e^{i\theta}, 1, \ldots, 1)x$, where $x \in U(N)$, $0 \leq \theta < 2\pi$, λ is normalized Haar measure on $U(N)$, and $C_a = \left(\int_0^{2\pi} (\sin(\theta/2))^a d\theta \right)^{-1}$. We take a to be an integer between 0 and $N-1$.

Let μ_k be the distribution of this random walk after k steps (where the starting distribution μ_0 is a point mass at the identity element of $U(N)$). We are interested in the convergence of μ_k to λ.

The case $a = N-1$ was studied in Porod's thesis, and it was proved that $\|\mu_k - \lambda\|_{L^2(\lambda)} \leq Ae^{-Bc}$ when $k = \frac{1}{2}N \log N + cN$, where A and B are positive constants. Then since $\|\mu_k - \lambda\|_{T.V.} \leq \frac{1}{2}\|\mu_k - \lambda\|_{L^2(\lambda)}$, her results implied convergence rates (in fact a cut-off phenomenon!) in total variation distance.

We have now observed that when $a = N-1$, μ_k in fact has a density in $L^2(\lambda)$ for $k \geq O(N)$.

Porod also showed that when $a = 0$, $\|\mu_k - \lambda\|_{L^2(\lambda)}$ was infinite for $k < \frac{1}{2}(N^2 - N) + 1$. This dramatically different behaviour prompted the present study, whose goal is to understand the L^2 convergence for intermediate values of a. In particular, we are interested in conditions on k as a function of a and N which would guarantee that μ_k is in $L^2(\lambda)$.

Using Fourier analysis and computing characters in a manner similar to the computation in Porod's thesis, we have shown that

---

* Dedicated to the memory of Onion Duck.

$$\|\mu_k - \lambda\|_{L^2(\lambda)} = K_{a,N,k} \sum_{\lambda_1<\lambda_2<\ldots<\lambda_N} \left(\sum_{\substack{j=1 \\ \lambda_j \le a}}^{N} (-1)^j \frac{\binom{-\lambda_j - a + N - 2}{N-2-2a}}{\prod_{r=j+1}^{N} (\lambda_r - \lambda_j) \prod_{r=1}^{j-1} (\lambda_j - \lambda_r)} \right)^{2k}$$

$$\times \left(\prod_{1 \le r < s \le N} (\lambda_s - \lambda_r) \right)^2 - 1 \,,$$

where $K_{a,N,k}$ is an explicit constant depending on a, N and k. Here the sum is taken over all N-tuples of (positive or negative) integers $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ satisfying $\lambda_1 < \lambda_2 < \ldots < \lambda_N$, and $\binom{-\lambda_j - a + N - 2}{N-2-2a}$ is a binomial coefficient.

We have further shown (by considering the sum over m of terms with $(\lambda_1, \ldots, \lambda_N) = (-m, m, 2m, \ldots, (N-1)m)$) that this sum is infinite for $k < (N^2 - N + 1)/2(a+1)$. (That is, μ_k is not a measure in $L^2(\lambda)$ for this range of k.)

(Andrey Feuerverger has now obtained similar lower bounds by related methods.)

On the other hand, a remark by Gerard Letac made us realize that, since the above measures are mutually absolutely continuous for different values of a, therefore since for $a = N - 1$ the measure μ_k is absolutely continuous with respect to λ for $k \ge O(N)$, therefore this same is true for any value of a.

The difficulties with further estimating the sum are that in the inside alternating sum, the individual terms may be going to infinite for large values of the λ_j, even though we know that the total value of the alternating sum is bounded by a constant. This makes analysis of the sum extremely sensitive.

One idea we had was to use spherical coordinates for $(\lambda_1, \ldots, \lambda_N)$, and to approximate the sum by an integral. Since we were only interested in the *finiteness* of the sum, we need only consider those $(\lambda_1, \ldots, \lambda_n)$ sufficiently far from the origin.