Gambling Systems and Multiplication-Invariant Measures

by

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1. Introduction.

This short paper describes a surprising connection between two previously unrelated topics: the probability of winning certain gambling games, and the invariance of certain measures under pointwise multiplication on the circle. The former has been well-studied by probabilists, notably in the book of Dubins and Savage (1965). The latter is the subject of an important and well-studied conjecture of H. Furstenberg. But to the best of our knowledge, the connection between them is new.

We shall show that associated with the gambling games are certain measures (whose cumulative distribution function values $F(x)$ are equal to the probability of winning the gambling game when starting with initial fortune x and using the strategy of "Bold Play"). We shall then show that these measures have interesting properties related to multiplication invariance; in particular, they give rise to a collection of measures which are invariant under multiplication by 2, and which have a weak limit which is also invariant under multiplication by 3. These measures provide some candidates for *possible* counterexamples to Furstenberg's conjecture.

Necessary background about gambling is presented in Section 2. Necessary background about multiplication-invariant measures is presented in Section 3. The connection between the two is discussed in Section 4. Some further observations are in Section 5.

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2. Gambling Games and Bold Play.

We shall consider gambling games which involve modifying our "fortune" by repeatedly making "bets". The goal of the game is to reach a fortune of 1 (in which case we win), before we reach a fortune of 0 (in which case we lose).

The rules of the gambling game are as follows. We begin with some initial fortune between 0 and 1. We also have a biased coin that comes up heads with probability p and tails with probability $1-p$ (for some $0<p<1$). We are further told some fixed positive number r (which represents the "payoff ratio").

We then repeatedly make bets as follows. If at some time we have a fortune x, then we may choose any value $y \le \min\{x,\ (1-x)/r\}$ as our next value to bet. (The choice of y may depend on the outcomes of previous bets, but not on the outcomes of future bets.) Given the bet value y, we flip the biased coin. If it comes up heads, we win and add ry to our fortune; if it comes up tails, we lose and subtract y from our fortune.

The game ends when we either reach a fortune of 1 (in which case we win), or reach a fortune of 0 (in which case we lose).

It is clear from this description that the probability of winning this game depends upon the *strategy* we employ, i.e. on how much we choose to bet for each turn. It was shown by Dubins and Savage (1965, pp. 90, 101; see also Billingsley, 1995, Theorem 7.3) that in the subfair case (i.e., when $rp < 1-p$), the probability of winning is maximized when the strategy employed is *Bold Play*. By Bold Play, we mean the strategy that, given fortune $x \in [0,1]$, chooses a bet value of $\min\{x,\ (1-x)/r\}$ (i.e., that bets the largest possible amount at each turn).

In the sequel, we shall write $F(x) = F_{r,p}(x)$ for the probability of winning the gambling game under the bold strategy, given parameters r and p and initial fortune x.

Remarks.

1. Clearly, in the superfair case (i.e., when $rp > 1-p$), Bold Play instead *minimizes* the probability of winning; this can be seen immediately by considering the game from the opponent's point of view. (In the fair case $rp = 1-p$, our strategy is irrelevant; all strategies give us probability x of winning, assuming only that with probability 1

the game eventually terminates.)

2. To see intuitively why Bold Play is the best strategy in the subfair case, note the following. If X_n is our fortune at time n, and B_n is the total amount bet up to time n, then $X_n - (pr - (1-p))B_n$ is a semi-bounded Martingale. Thus, letting $n \to \infty$, we see that for strategies which terminate with probability 1,

$$\mathbf{P}_x(\text{win}) \;=\; x + (pr - (1-p))\,\mathbf{E}_x(\text{total amount bet})\,.$$

In particular, for subfair games, maximizing the probability of winning is equivalent to minimizing the expected total bet.

3. Multiplication-Invariant Measures and Furstenberg's Conjecture.

We let $\tau_n : [0,1] \to [0,1]$ denote the function $\tau_n(x) = nx \bmod 1$; and for a Borel measure μ on [0,1), we let $\mu \circ \tau_n$ be defined by $\mu \circ \tau_n(a,b] = \int 1_{(a,b]}(\tau_n(x))d\mu(x)$.

These functions are related to a conjecture of H. Furstenberg. Say that μ is "$\times n$ invariant" if $\mu \circ \tau_n = \mu$, i.e. if the measure is unchanged upon multiplying the circle by a factor of n. Furstenberg conjectured that any probability measure on $[0,1)$ which is simultaneously both $\times 2$ and $\times 3$ invariant must be a convex combination of Lebesgue measure and a purely atomic measure. This conjecture has received a great deal of attention (Furstenberg, 1967; Lyons, 1988; Rudolph, 1990; Feldman and Smordinsky, 1992; Feldman, 1993). It particular, it has been shown that under additional hypotheses the conjecture is true. However the original conjecture remains unsolved.

In what follows, we shall use the gambling games to construct a measure which is simultaneously both $\times 2$ and $\times 3$ invariant, but which is not obviously a combination of Lebesgue and atomic measures. If it could be shown definitely to *not* be a convex combination of Lebesgue measure and an atomic measure, then that would provide a counterexample to Furstenberg's conjecture. (Unfortunately we are unable to show this.)

4. Gambling Measures and $\times 2 \times 3$ Invariance.

We recall that $F(x) = F_{r,p}(x)$ stands for the probability of winning the gambling game under the bold strategy, with parameters r and p and initial fortune x. We note that $F(0) = 0$, that $F(1) = 1$, and that F is non-decreasing on $[0, 1]$. Furthermore, it is straightforward to show that F is a continuous function. Thus, we can define probability measures $\mu = \mu_{r,p}$ by the formula $\mu(a, b] = F(b) - F(a)$. It is these "gambling measures" which will provide the connection to Furstenberg's conjecture. A key computation is

Proposition 1. *For the case where $r = n$ is a positive integer, we have that*

$$\mu_{n,p} \circ \tau_{n+1} \;=\; p\, \mu_{n,p} + (1-p)(\mu_{n,p} \circ \tau_n)\,.$$

In words, composing the measure $\mu_{n,p}$ with the function τ_{n+1} results in a convex combination of the measure itself, and the measure composed with τ_n.

Proof. Set $F(x) = \mu_{n,p}[0, x]$. Elementary arguments show that the proposition is equivalent to the following equation involving F:

$$\sum_{j=0}^{n} \left(F(\frac{x+j}{n+1}) - F(\frac{j}{n+1}) \right) \;=\; pF(x) + (1-p) \sum_{j=0}^{n-1} \left(F(\frac{x+j}{n}) - F(\frac{j}{n}) \right) \qquad (\dagger)$$

Now, given a current fortune x there are two possibilities arising from the alternatives in Bold Play. We see by inspection that if $x < \frac{1}{n+1}$, then $F(x) = pF((n+1)x)$. Similarly, if $x > \frac{1}{n+1}$, then $F(x) = p + (1-p)F(x - \frac{1-x}{n}) = p + (1-p)F(\frac{(n+1)x-1}{n})$.

These observations imply that for any $x \in [0, 1]$, we have $F(\frac{x}{n+1}) = pF(x)$ and $F(\frac{x+j}{n+1}) - F(\frac{j}{n+1}) = (1-p)(F(\frac{x+j-1}{n}) - F(\frac{j-1}{n}))$ for $j = 1, 2, ...n$. Summing these equations over j establishes ($\dagger$), and hence also establishes the proposition. ∎

In particular, applying this proposition with $n = 1$, we see that $\mu_{1,p} \circ \tau_2 = \mu_{1,p}$, i.e. $\mu_{1,p}$ is $\times 2$ invariant (for any $0 < p < 1$). Unfortunately the measures $\mu_{1,p}$ are not also $\times 3$ invariant (unless $p = 1/2$, in which case we obtain Lebesgue measure).

Applying the proposition with $n = 2$, we see that $\mu_{2,p} \circ \tau_3 = p\mu_{2,p} + (1-p)(\mu_{2,p} \circ \tau_2)$. It follows that <u>if</u> $\mu_{2,p}$ were $\times 2$ invariant, then it would also be $\times 3$ invariant (and hence a candidate for a counterexample to Furstenberg's conjecture). Unfortunately this is not the case; for example it is straightforward to show that $\mu_{2,1/2}[0,\frac{1}{6}] \neq (\mu_{2,1/2} \circ \tau_2)[0,\frac{1}{6}]$.

However, note that since $(\nu \circ \tau_m) \circ \tau_n = (\nu \circ \tau_n) \circ \tau_m$, we have that $\nu \circ \tau_3 = p\,\nu + (1-p)\,(\nu \circ \tau_2)$, whenever $\nu = \mu_{2,p} \circ \tau_k$ for *any* positive integer k. It follows immediately that this equation also holds when ν is any weak* limit of any convex combination of these measures. This suggests that we try to find such a measure which is also $\times 2$ invariant. For concreteness we shall focus on the case $p = 1/2$.

Lemma 2. *We have that* $\mu_{2,\frac{1}{2}} \circ \tau_{3^m} = \frac{1}{2^m} \sum_{j=0}^m \binom{m}{j} (\mu_{2,\frac{1}{2}} \circ \tau_{2^j})$

Proof. The previous proposition proves the case where $m = 1$. As an inductive hypothesis assume the formula is established for all $m \leq M$. For convenience denote by μ the measure $\mu_{2,\frac{1}{2}}$. Using the inductive hypothesis and applying the case $m = 1$ to the measures $\mu \circ \tau_{2^j} \circ \tau_3 = \mu \circ \tau_3 \circ \tau_{2^j}$, we compute as follows:

$$\mu \circ \tau_{3^{M+1}} = \frac{1}{2^M} \sum_{j=0}^{M} \binom{M}{j} \frac{1}{2} \left((\mu \circ \tau_{2^k}) + (\mu \circ \tau_{2^{k+1}}) \right)$$

$$= \frac{1}{2^{M+1}} \left(\mu + (\mu \circ \tau_{2^{M+1}}) + \sum_{j=1}^{M} \left[\binom{M}{j} + \binom{M}{j-1} \right] (\mu \circ \tau_{2^j}) \right)$$

$$= \frac{1}{2^{M+1}} \sum_{j=0}^{M+1} \binom{M+1}{j} (\mu \circ \tau_{2^j}),$$

since $\binom{M}{j} + \binom{M}{j-1} = \binom{M+1}{j}$. Thus the formula holds for $m = M+1$ and the lemma is proved. ∎

Proposition 3. *Fix $r = 2$ and $p = \frac{1}{2}$. Then* $\lim_{m\to\infty} \|\mu_{2,1/2} \circ \tau_{3^m} - \mu_{2,1/2} \circ \tau_{2\cdot 3^m}\| = 0$, *where $\|\ldots\|$ denotes total variation distance. (That is, the measures $\mu_{2,1/2} \circ \tau_{3^m}$ are asymptotically $\times 2$ invariant, as $m \to \infty$.)*

Proof. Using the previous lemma, noting that $\mu_{2,1/2} \circ \tau_{2\cdot 3^m} = (\mu_{2,1/2} \circ \tau_{3^m}) \circ \tau_2$, and recalling that $\| \ldots \|$ is a metric, we see that

$$\begin{aligned}
\|\mu_{2,1/2} \circ \tau_{3^m} - \mu_{2,1/2} \circ \tau_{2\cdot 3^m}\| &\leq \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} \|\mu_{2,1/2} \circ \tau_{2^j} - \mu_{2,1/2} \circ \tau_{2^{j+1}}\| \\
&\leq \frac{2}{2^m} + \frac{1}{2^m} \sum_{j=1}^{m} \left| \binom{m}{j} - \binom{m}{j-1} \right| \\
&= \frac{2}{2^m} + \frac{1}{2^m} \sum_{j=1}^{m} \frac{1}{m+1} \binom{m+1}{j} |m+1-2j| \\
&= \frac{1}{2^m} \sum_{j=0}^{m+1} \binom{m+1}{j} \left| 1 - \frac{2j}{m+1} \right| .
\end{aligned}$$

Now, this expression is simply twice the expected value of $|1 - \frac{2X}{m+1}|$, where X is a random variable having distribution Binomial$(m+1,\ 1/2)$. But then $\mathbf{E}(X) = \frac{1}{2}(m+1)$, whence $\mathbf{E}\left(1 - \frac{2X}{m+1}\right) = 0$. Furthermore $1 - \frac{2X}{m+1}$ is bounded, and may be represented as the average of $m+1$ different mean-0 i.i.d. terms, viz.

$$1 - \frac{2X}{m+1} = \frac{1}{m+1} \sum_{k=1}^{m+1} (1 - 2X_k) ,$$

with $\{X_k\}$ i.i.d. $\sim$ Bernoulli(1/2). Hence, by the strong law of large numbers (see e.g. Billingsley, 1995, Theorem 22.1) we have $1 - \frac{2X}{m+1} \to 0$ as $m \to \infty$. Then by the bounded convergence theorem (see e.g. Billingsley, 1995, Theorem 16.5), we have that $\mathbf{E}\left|1 - \frac{2X}{m+1}\right| \to 0$ as $m \to \infty$. Thus our upper bound converges to 0, and the proposition is shown. ∎

Now, the set of all probability measures are compact under the weak* topology. Hence, there exists at least one weak* limit point of the measures $\{\mu_{2,1/2} \circ \tau_{3^m}\}$. If ν is such a measure, then we necessarily have that $\nu \circ \tau_3 = p\,\nu + (1-p)\,(\nu \circ \tau_2)$. Furthermore, by the above proposition, we also have that $\nu \circ \tau_2 = \nu$, i.e. that ν is $\times 2$ invariant. We thus obtain

Theorem 4. *The exists at least one weak* limit point of the set of measures* $\{\mu_{2,1/2} \circ \tau_{3^m}\}_{m=1}^{\infty}$*; and all such limits are simultaneously both* $\times 2$ *and* $\times 3$ *invariant. Furthermore,* <u>*if*</u> *one such limit is not a convex combination of Lebesgue measure and a purely atomic measure, then this measure provides a counterexample to Furstenberg's conjecture.*

Unfortunately, we are unable to establish the existence of such a counterexample. We believe that any such weak* limit would be non-atomic; however it is quite possible that all such weak* limits are simply Lebesgue measure on $[0, 1]$.

5. Further Properties of the Measures $\mu_{r,p}$.

The measures $\mu_{r,p}$ are interesting in their own right. We make a few further observations about them here.

We shall have occasion to consider Bold Play on the interval $[a, b]$ (where $0 \leq a < b \leq 1$). By this we mean the betting strategy that, given fortune $x \in [a, b]$, chooses a bet value of $\min\{x - a, \frac{b-x}{r}\}$; and then repeats this process until either the fortune a or the fortune b is achieved. For $x \in [a, b]$, let $F_{[a,b]}(x)$ denote the probability of increasing our fortune from x to b, under Bold Play on the interval $[a, b]$.

We observe that if $\frac{x-a}{b-a} = \frac{x'-a'}{b'-a'}$, then clearly $F_{[a',b']}(x') = F_{[a,b]}(x)$. We shall use this in our calculations below.

We next observe that if $x \in [0, \frac{1}{r+1}]$, then ordinary Bold Play (i.e., on $[0, 1]$) is *equivalent* (in the sense of giving the same overall probability of winning the gambling game) to Bold Play on $[0, \frac{1}{r+1}]$ followed by Bold Play on $[0, 1]$. This fact is well known (cf. Dubins and Savage, 1965), and in any case follows from the observation that for $0 \leq x \leq \frac{1}{r+1}$, we have $F_{[0,1]}(x) = pF_{[0,1]}((r + 1)x) = pF_{[0,\frac{1}{r+1}]}(x)$. Similarly, if $x \in [\frac{1}{r+1}, 1]$ then ordinary Bold Play is equivalent to Bold Play on $[\frac{1}{r+1}, 1]$ followed by Bold Play on $[0, 1]$; this follows simply by considering the game from the opponent's point of view.

We can inductively apply this observation, as follows. Let S_0 and S_1 be two operators which act on intervals, by keeping the first $\frac{1}{r}$ or the last $\frac{r}{r+1}$ of the interval, respectively. That is,

$$S_0[a, b) \;=\; [a,\; a + \frac{b-a}{r+1}) \,; \qquad S_1[a, b) \;=\; [a + \frac{b-a}{r+1},\; b) \,.$$

By induction, we have the following.

Proposition 5. *Let I be an interval of the form*

$$I = S_{a_1} S_{a_2} \ldots S_{a_n} [0,1)\,,$$

for some $n \in \mathbf{N}$ and some $a_1, a_2, \ldots, a_n \in \{0,1\}$. Let $x \in I$. Then the strategy of using Bold Play on I, followed by Bold Play on $[0,1]$, is equivalent to the strategy of ordinary Bold Play on $[0,1]$.

Remark. By applying the lemma repeatedly, we see that it is also equivalent to (say) apply Bold Play first on I, then on $S_{a_1} S_{a_2} S_{a_3} [0,1)$, then on $S_{a_1}[0,1)$, and then on $[0,1]$. Indeed, any nested sequence of intervals of the form $S_{a_1} \ldots S_{a_k}[0,1)$ may be used.

The above lemma suggests writing numbers $x \in [0,1)$ in terms of their r-ary expansion, by which we mean the (unique) sequence $\{a_i\}_{i=1}^{\infty}$, with $a_i \in \{0,1\}$ for each i, such that $x \in S_{a_1} S_{a_2} \ldots S_{a_n}[0,1)$ for all $n \in \mathbf{N}$. (For $x = 1$, we instead assign the special r-ary expansion $a_1 = a_2 = \ldots = 1$.) Equivalently, this means that

$$x = \sum_{i=1}^{\infty} \frac{1}{r+1}\, a_i \prod_{j=1}^{i-1} \left(\frac{1}{r+1} + \frac{r-1}{r+1}\, a_j \right)$$

(where we take $\prod_{j=1}^{0} (\ldots) = 1$ if it occurs).

Such expansions are related to betting using Bold Play. Specifically, we have the following proposition (whose proof we omit).

Proposition 6. *Let $x \in [0,1]$ have r-ary expansion $\{a_i\}_{i=1}^{\infty}$, as defined above. Let $F_{r,p}(x)$ denote the probability of winning the gambling game with parameters r and p, with initial fortune x. Then*

$$F_{r,p}(x) = \sum_{i=1}^{\infty} p\, a_i \prod_{j=1}^{i-1} (p + (1-2p) a_j)\ .$$

That is, we can compute $F_{r,p}(x)$ by writing x in its r-ary expansion, and then evaluating the resulting sequence as a $\left(\frac{1-p}{p}\right)$-ary expansion.

Remark. We note that $\frac{1-p}{p} = r$ precisely for fair games, in which case $F_{r,p}(x) = x$ (as it must).

This proposition is related to (and, indeed, is implied by) another way of constructing the measures $\mu_{r,p}$. Define an operator $T_{r,p}$, acting on measures λ on $[0,1]$, by

$$(T_{r,p}\lambda)[0,x] = \begin{cases} p\,\lambda[0,(r+1)x]\,, & x < \dfrac{1}{r+1}\,; \\ p \;+\; (1-p)\,\lambda[0,\; x - \dfrac{1-x}{r}]\,, & x \geq \dfrac{1}{r+1}\,. \end{cases}$$

Then we have the following (we again omit the proof).

Proposition 7. *The measure $\mu_{r,p}$ is invariant under $T_{r,p}$, i.e. $T_{r,p}\,\mu_{r,p} = \mu_{r,p}$. Furthermore, as $n \to \infty$, we have for any non-atomic measure λ that $T^n_{r,p}\lambda \to \mu_{r,p}$, in the sense that*

$$\lim_{n\to\infty} \sup_{0\leq a<b\leq 1} \left|T^n_{r,p}\lambda(a,b] \;-\mu_{r,p}(a,b]\right| \;=\; 0\,,$$

i.e. the corresponding cumulative distribution functions converge uniformly. (This is stronger than weak convergence, but weaker than convergence in total variation distance.)*

In light of this proposition, the measures $\mu_{r,p}$ may be seen as special cases of measures whose Fourier transforms satisfy curious functional equations. These measures are obtained by partitioning the interval into finitely many pieces according to some fixed procedure while performing some fixed action on each piece (like our transformation $T_{r,p}$ above); and then continuing inductively on each piece. It is easily seen that by this method, for any $\{\alpha_i\}$ and $\{n_i\}$ with $\alpha_i \geq 0$ and $n_i \in \mathbf{N}$, satisfying $\sum \alpha_i = 1$ and $\sum n_i \leq n$, one can construct measures μ whose Fourier transforms $\widehat{\mu}$ satisfy

$$\sum_i \alpha_i \widehat{\mu}(n_i) = \widehat{\mu}(n)\,.$$

Some related matters are considered by de Rham (1956–57).

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