A Clarification in the Paper eigen.pdf

This note expands on the argument at the top of page 11 of the paper "Convergence rates of Markov chains" (J.S. Rosenthal, SIAM Review **37** (1995), 387–405).

As stated there, suppose P is the transition matrix for a Markov chain on a finite state space $\mathcal{X}$, and $x \in \mathcal{X}$ is transient, and we have found $y \in \mathcal{X}$ and $r \in \mathbf{N}$ such that $P^r(x,y) = \epsilon > 0$ but $P^m(y,x) = 0$ for all $m \geq 0$.

Then if $T = \{j \in \mathcal{X} : P^m(j,x) > 0 \text{ for some } m \geq 0\}$, then $x \in T$ but $y \notin T$.

Furthermore, for $j \in T$ and $z \notin T$, we must have $P^n(z,j) = 0$ for all n. [Proof: Since $j \in T$, we must have $P^m(j,x) > 0$ for some $m \geq 0$. If we had $P^n(z,j) > 0$, then we would have $P^{n+m}(z,x) \geq P^n(z,j)\,P^m(j,x) > 0$, contradicting the fact that $z \notin T$.]

Using this fact, we then compute that

$$\sum_{j \in T} |(vP^r)(j)| \;=\; \sum_{j \in T}\sum_{z \in \mathcal{X}} |v(z)\,P^r(z,j)| \;=\; \sum_{j \in T}\sum_{z \in T} |v(z)\,P^r(z,j)|\;.$$

Hence,

$$\begin{aligned}
\sum_{j \in T} |(vP^r)(j)| \;&=\; \Big(\sum_{z \in T}\sum_{j \in \mathcal{X}} |v(z)\,P^r(z,j)|\Big) - \Big(\sum_{z \in T}\sum_{j \notin T} |v(z)\,P^r(z,j)|\Big) \\
&\leq\; \Big(\sum_{z \in T}\sum_{j \in \mathcal{X}} |v(z)|\;P^r(z,j)\Big) - \Big(\sum_{z \in T}\sum_{j \notin T} |v(z)\,P^r(z,j)|\Big) \\
&=\; \Big(\sum_{z \in T} |v(z)|\Big) - \Big(\sum_{z \in T}\sum_{j \notin T} |v(z)\,P^r(z,j)|\Big)\,.
\end{aligned}$$

Then, since $x \in T$ but $y \notin T$, it follows as claimed that

$$\sum_{j \in T} |(vP^r)(j)| \;\leq\; \Big(\sum_{z \in T} |v(z)|\Big) - \Big(|v(x)\,P^r(x,y)|\Big) \;=\; \Big(\sum_{z \in T} |v(z)|\Big) - \epsilon\;|v(x)|\;.$$

Then, as noted in the paper, if $vP = \lambda v$ with $|\lambda| = 1$, then

$$\sum_{j \in T} |(vP^r)(j)| \;=\; \sum_{j \in T} |\lambda^r\,v(j)| \;=\; \sum_{j \in T} |v(j)|\;,$$

so we must have $\epsilon\,|v(x)| = 0$, hence $v(x) = 0$, as claimed.