Maximum Binomial Probabilities and Game Theory Voter Models

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This note is motivated by questions in voting game theory, which concerns itself with models of how people decide to vote (see e.g. [2, 3, 5] and many other references). A simple voter model assumes there are two candidates A and B, with n_A and n_B supporters respectively who each vote independently with probabilities p_A and p_B respectively. Under such assumptions, what is the probability that the vote ends in a tie, or within some fixed margin α? Such questions are directly related to bounds on maximum binomial probabilities (Theorem 1), which in turn allow us to bound probabilities of differences of pairs of independent variables (Corollary 5), and of vote margins (Corollary 6). We also extend our results to assumptions involving the vote size (Propositions 7 and 8 and Corollaries 9, 10, and 11).

To state our results, let $f(n, p; k)$ be the probability that a binomial distribution with parameters n and p equals the specific value k, i.e.

$$f(n, p; k) \ := \ \mathbf{P}[\text{Binomial}(n, p) = k] \ = \ \binom{n}{k} p^k \, (1-p)^{n-k} \ = \ \frac{n!}{k! \, (n-k)!} \, p^k \, (1-p)^{n-k} \, .$$

The *mode* of f is well-known (e.g. [4], p. 70), but the *maximal* values of f are less well studied. Note that $f(n, 0; 0) = f(n, 1; n) = 1$ for any $n \in \mathbf{N}$, so there are no non-trivial upper bounds in general. However, if p is bounded away from 0 and 1, then we prove:

Theorem 1 *To first order as $n \to \infty$, the maximum binomial probability over all $\epsilon \le p \le 1-\epsilon$ equals $\frac{1}{\sqrt{2\pi n \epsilon(1-\epsilon)}}$. More precisely, for any fixed $\epsilon \in (0, 1/2]$,*

$$\lim_{n\to\infty} \frac{\sup_{\epsilon \le p \le 1-\epsilon} \ \max_{0 \le k \le n} \ f(n, p; k)}{\frac{1}{\sqrt{2\pi n \epsilon(1-\epsilon)}}} \ = \ 1 \, .$$

In particular, $\lim_{n\to\infty} \sup_{\epsilon \le p \le 1-\epsilon} \max_{0 \le k \le n} f(n, p; k) \to 0$.

We begin the proof of Theorem 1 with some lemmas.

Lemma 2 *Let $T_{n,p} = \frac{1}{n} \lfloor (n+1)p \rfloor$. Then $T_{n,p} \approx p$, and $f(n, p; \cdot)$ is unimodal with mode at $nT_{n,p}$. Specifically:* **(a)** $|T_{n,p} - p| \le \frac{1}{n}$, *and* **(b)** $\max_k f(n, p; k) = f(n, p; nT_{n,p})$, *and* **(c)** *if $0 \le k_1 \le k_2 \le nT_{n,p}$ or $n \ge k_1 \ge k_2 \ge nT_{n,p}$, then $f(n, p; k_1) \le f(n, p; k_2)$.*

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Proof. Part (a) is trivial, and parts (b) and (c) follow since $f(n,p;k+1)\ /\ f(n,p;k) = (n-k)p\ /\ (k+1)(1-p)$, which is > 1 for $k < nT_{n,p}$ and < 1 for $k > nT_{n,p}$. ∎

Remark. If $(n+1)p$ is an integer, then there are actually two adjacent modes, but the one at $nT_{n,p}$ suffices for our purposes.

Lemma 3 *For any $0 < p < 1$, as n and k and $n-k$ all $\to \infty$,*

$$f(n,p;k) \;=\; \Big[g_p(k/n)\Big]^n \sqrt{1/2\pi k[1-(k/n)]}\;\Big[1+o(1)\Big],$$

where

$$g_p(t) \;=\; \left(\frac{p}{t}\right)^t \left(\frac{1-p}{1-t}\right)^{1-t}, \quad t \in (0,1)\,.$$

Proof. Recall Stirling's Approximation (e.g. [1], p. 113): as $n \to \infty$, $n! = (n/e)^n\sqrt{2\pi n}\,[1+o(1)]$. Hence, for any $0 < p < 1$, as n and k and $n-k$ all $\to \infty$,

$$f(n,p;k) \;=\; \frac{(n/e)^n\sqrt{2\pi n}\;\; p^k(1-p)^{n-k}}{(k/e)^k\sqrt{2\pi k}\;[(n-k)/e]^{n-k}\sqrt{2\pi(n-k)}}\;\Big[1+o(1)\Big]\,.$$

After some cancellation, this gives that

$$\begin{aligned} f(n,p;k) \;&=\; \left(\frac{p}{k/n}\right)^k \left(\frac{1-p}{1-(k/n)}\right)^{n-k} \sqrt{1/2\pi k[1-(k/n)]}\;\Big[1+o(1)\Big] \\ &=\; \Big[g_p(k/n)\Big]^n \sqrt{1/2\pi k[1-(k/n)]}\;\Big[1+o(1)\Big]\,. \end{aligned}$$ ∎

Lemma 4 *Fix $p \in (0,1)$, and let $g_p(t)$ be as in Lemma 3. Then* **(a)** *$g_p(t) \le 1$ for all $t \in (0,1)$, and* **(b)** *$g_p(p+r) = 1 + O(r^2)$ as $r \to 0$.*

Proof. We compute that

$$\log g_p(t) = t\log(p) - t\log(t) + (1-t)\log(1-p) - (1-t)\log(1-t)\,.$$

Hence,

$$\frac{d}{dt}\log(g_p(t)) \;=\; \log(p) - \log(t) - \log(1-p) + \log(1-t) \;=\; \log(\frac{p}{1-p}) - \log(\frac{t}{1-t})\,.$$

This equals 0 when and only when $t = p$. Furthermore

$$\left(\frac{d}{dt}\right)^2 \log(g_p(t)) \;=\; -\frac{1}{t} - \frac{1}{1-t} \;=\; -\frac{1}{t(1-t)} \;\le\; -4 \;<\; 0\,.$$

It follows that $\log g_p$, and hence also g_p, achieves its maximum when $t = p$. Hence, for all $0 < t < 1$, we have $g_p(t) \le g_p(p) = 1$. Then, taking a Taylor expansion around $t = p$ gives

$$\begin{aligned}\log g_p(p+r) &= \log g_p(p) + r\frac{d}{dt}\log g_p(t)\Big|_{t=p} + \frac{r^2}{2}\left(\frac{d}{dt}\right)^2 \log g_p(t)\Big|_{t=p} + O(r^3) \\ &= 0 + r(0) + \frac{r^2}{2}\left(-\frac{1}{p} - \frac{1}{1-p}\right) + O(r^3) = O(r^2).\end{aligned}$$

Hence, $g_p(p+r) = \exp[O(r^2)] = 1 + O(r^2)$, as claimed. ∎

Remark. It follows from the proof of Lemma 4 that $g_p(t) \le \exp(-2(t-p)^2)$ for all $t \in (0,1)$, though we do not use that fact here.

Proof of Theorem 1. Since $|T_{n,p} - p| = O(1/n)$ by Lemma 2(a), it follows from Lemmas 3 and 4(b) that as $n \to \infty$,

$$\begin{aligned}f(n,p;nT_{n,p}) &= \Big[g_p(T_{n,p})\Big]^n \sqrt{1/2\pi n T_{n,p}[1-T_{n,p}]}\,\Big[1+o(1)\Big] \\ &= \Big[g_p\Big(p + O(1/n)\Big)\Big]^n \sqrt{1/2\pi n T_{n,p}[1-T_{n,p}]}\,\Big[1+o(1)\Big] \\ &= \Big[1 + O(1/n^2)\Big]^n \sqrt{1/2\pi n T_{n,p}[1-T_{n,p}]}\,\Big[1+o(1)\Big] \\ &= e^{nO(1/n^2)}\sqrt{1/2\pi np[1-p]}\,\Big[1+o(1)\Big]. \\ &= \sqrt{1/2\pi np[1-p]}\,\Big[1+o(1)\Big].\end{aligned}$$

Hence, using Lemma 2(b),

$$\begin{aligned}\sup_{\epsilon\le p\le 1-\epsilon}\max_{0\le k\le n} f(n,p;k) &= \sup_{\epsilon\le p\le 1-\epsilon} f(n,p;nT_{n,p}) \\ &= \sup_{\epsilon\le p\le 1-\epsilon}\sqrt{1/2\pi np(1-p)}\,\Big[1+o(1)\Big] \\ &= \sqrt{1/2\pi n\epsilon(1-\epsilon)}\,\Big[1+o(1)\Big],\end{aligned}$$

which gives the result. ∎

Remark. The Central Limit Theorem (CLT) says that the Binomial(n,p) distribution can be approximated by a normal distribution with mean $m = np$ and variance $v = np(1-p)$, with density function $\frac{1}{\sqrt{2\pi v}}e^{-(x-m)^2/2v}$ and hence maximal density value $\frac{1}{\sqrt{2\pi v}} = \frac{1}{\sqrt{2\pi np(1-p)}}$. The CLT does not directly imply maximum probabilities, but this maximal density value is consistent with the maximum probabilities in the proof of Theorem 1.

Theorem 1 has implications for pairs of independent binomial random variables:

Corollary 5 *Let X and Y be two independent random variables having binomial distributions with parameters n_X, p_X and n_Y, p_Y, respectively. Then for any $\epsilon \in (0, 1/2]$ and $\alpha < \infty$,*

$$\lim_{n_X \to \infty} \sup_{\epsilon \le p_X \le 1-\epsilon} \sup_{n_Y \in \mathbf{N}} \sup_{0 \le p_Y \le 1} \mathbf{P}(|X - Y| \le \alpha) \;=\; 0\,.$$

That is, for large n_X, if p_X is bounded away from 0 and 1, then the probability that X and Y are within any fixed tolerance α goes to zero regardless of the values of n_Y and p_Y.

Proof. Here

$$\begin{aligned}
\mathbf{P}(|X - Y| \le \alpha) \;&=\; \sum_z \sum_{|d| \le \alpha} \mathbf{P}[Y = z,\, X = z + d] \\
&=\; \sum_z \sum_{|d| \le \alpha} \mathbf{P}[X = z + d]\; \mathbf{P}[Y = z] \\
&\le\; \sum_z \sum_{|d| \le \alpha} (\sup_w \mathbf{P}[X = w])\; \mathbf{P}[Y = z] \\
&\le\; (2\alpha + 1)\; \sup_w \mathbf{P}[X = w]\,,
\end{aligned}$$

and this last quantity goes to 0 as $n_X \to \infty$ by Theorem 1. ∎

Then, putting this in the context of voting theory, we conclude:

Corollary 6 *Suppose n_A voters each independently vote for candidate A with probability p_A (otherwise they don't vote), and similarly n_B and p_B for candidate B. Let X and Y be the total votes received by candidates A and B, respectively, so $X \sim \mathrm{Binomial}(n_A, p_A)$ and $Y \sim \mathrm{Binomial}(n_B, p_B)$ are independent. Then for any $\epsilon \in (0, 1/2]$ and $\alpha < \infty$,*

$$\lim_{n_A \to \infty} \sup_{\epsilon \le p_A \le 1-\epsilon} \sup_{n_B \in \mathbf{N}} \sup_{p_B \in \mathbf{R}} \mathbf{P}(|X - Y| \le \alpha) \;=\; 0\,.$$

That is, as n_A goes to infinity, if the corresponding vote probability p_A is bounded away from 0 and 1, then the probability that the two vote counts are within any fixed finite tolerance α of each other goes to 0, regardless of the values of n_B and p_B.

We can also extend Theorem 1 to restrictions on k instead of p:

Proposition 7 *For any fixed $r \in (0, 1/2]$, as $n \to \infty$,*

$$\sup_{0 \le p \le 1} \max_{rn \le k \le (1-r)n} f(n, p; k) \;=\; \frac{1}{\sqrt{2\pi n r(1-r)}}\,[1 + o(1)] \;\to\; 0\,.$$

Proof. We consider different ranges of p separately.

For $p = 0$ or $p = 1$, clearly $\max_{rn \le k \le (1-r)n} \; f(n,p;k) = 0$.

For $p \in [r,\, 1-r]$, we have by Theorem 1 that

$$\max_{rn \le k \le (1-r)n} \; f(n,p;k) \;\le\; \sup_{r \le p' \le 1-r} \; \max_{0 \le k \le n} \; f(n,p';k) \;=\; \frac{1}{\sqrt{2\pi n r(1-r)}}[1+o(1)]\,,$$

with equality when $p = r$ [and when $p = 1-r$].

For $0 < p < r$, it follows from Lemma 2(c) and Lemma 3 that

$$\max_{rn \le k \le (1-r)n} \; f(n,p;k) \;=\; f(n,p;\lceil rn \rceil) \;=\; \Big[g_p(r)\Big]^n \sqrt{1/2\pi n r(1-r)}\; \Big[1+o(1)\Big]\,.$$

Hence, by Lemma 4 part (a),

$$\max_{rn \le k \le (1-r)n} \; f(n,p;k) \;\le\; \sqrt{1/2\pi n r(1-r)}\; \Big[1+o(1)\Big]\,.$$

Similarly, for $1-r < p < 1$,

$$\max_{rn \le k \le (1-r)n} \; f(n,p;k) \;\le\; \sqrt{1/2\pi n r(1-r)}\; \Big[1+o(1)\Big]\,.$$

This covers all possible values of p, so the result follows. ∎

Or, instead, to a "mix" of restrictions on p and on k:

Proposition 8 *For any fixed $\epsilon, r \in (0, 1/2]$, setting $m = \min(\epsilon,\, r)$, as $n \to \infty$,*

$$\sup_{\epsilon \le p \le 1} \; \max_{0 \le k \le (1-r)n} \; f(n,p;k) \;=\; \sup_{0 \le p \le 1-\epsilon} \; \max_{rn \le k \le n} \; f(n,p;k) \;=\; \frac{1}{\sqrt{2\pi n m(1-m)}}\,[1+o(1)] \;\to\; 0\,.$$

Proof. We focus on the first quantity $\sup_{\epsilon \le p \le 1} \; \max_{0 \le k \le (1-r)n} \; f(n,p;k)$; the proof for the second quantity is symmetric. It follows from Lemma 3 that equality is achieved when $p = m$ and $k = \lfloor mn \rfloor$ if $\epsilon \le r$, or when $p = 1-m$ and $k = \lfloor (1-m)n \rfloor$ if $r \le \epsilon$. To show that no larger value can arise, we again consider different ranges of $p \in [\epsilon, 1]$ separately.

For $p = 1$, clearly $\max_{0 \le k \le (1-r)n} \; f(n,p;k) = 0$.

For $p \in [\epsilon,\, 1-m] \subseteq [m,\, 1-m]$, by Theorem 1,

$$\max_{0 \le k \le (1-r)n} \; f(n,p;k) \;\le\; \sup_{m \le p' \le 1-m} \; \max_{0 \le k \le n} \; f(n,p';k) \;=\; \frac{1}{\sqrt{2\pi n m(1-m)}}[1+o(1)]\,.$$

For $p \in (1-m,\, 1)$, it follows from Lemma 2(c) and Lemma 3 and Lemma 4(a) that

$$\max_{0 \le k \le (1-r)n} \; f(n,p;k) \;=\; f(n,p;\lfloor (1-r)n \rfloor) \;=\; \Big[g_p(1-r)\Big]^n \sqrt{1/2\pi n r(1-r)}\; \Big[1+o(1)\Big]$$

$$\le\; \sqrt{1/2\pi n r(1-r)}\; \Big[1+o(1)\Big] \;\le\; \sqrt{1/2\pi n m(1-m)}\; \Big[1+o(1)\Big]$$

since $0 < m \le r \le 1/2$ implies that $r(1-r) \ge m(1-m)$. This covers all $p \in [\epsilon, 1]$. ∎

In the context of voting theory, Proposition 8 gives:

Corollary 9 *In the setup of Corollary 6, for any $\epsilon, r \in (0,1)$,*

$$\lim_{n_A \to \infty} \sup_{p_A \geq \epsilon} \sup_{n_B \leq (1-r)n_A} \sup_{0 \leq p_B \leq 1} \mathbf{P}(|X-Y| \leq \alpha) \;=\; 0\,.$$

That is, as n_A goes to infinity, if the vote probability p_A is bounded away from 0 (but not 1), and the size n_B is no more than a fraction < 1 of n_A, then the probability that the two vote counts are within any fixed finite tolerance α of each other still goes to 0.

Proof. Here

$$\mathbf{P}(|X-Y| \leq \alpha) \;=\; \sum_z \sum_{|d| \leq \alpha} \mathbf{P}[Y=z,\, X=z+d]$$

$$\leq\; (2\alpha+1) \max_{0 \leq w \leq n_B} \mathbf{P}[X=w] \;\leq\; (2\alpha+1) \max_{0 \leq w \leq (1-r)n_A} \mathbf{P}[X=w]$$

which $\to 0$ as $n_A \to \infty$ by Proposition 8 (reducing ϵ and r to $\leq 1/2$ if necessary). ∎

Remark. In Corollaries 6 and 9, the restriction that p_A or n_B/n_A be bounded away from 1 really is necessary, even if $n_A \neq n_B$. For example, if $n_B = n_A - 1$ and $p_A = 1 - (1/n_A)$ and $p_B = 1$, then as $n_A \to \infty$,

$$\mathbf{P}(X=Y) \;\geq\; \mathbf{P}(X=Y=n_B) \;=\; \mathbf{P}(X=n_B)\,\mathbf{P}(Y=n_B) \;=\; \binom{n_A}{n_B} p_A^{n_B}(1-p_A)^1\;(1)$$

$$=\; n_A\,[1-(1/n_A)]^{n_B}(1/n_A) \;=\; e^{-n_B/n_A}\,[1+o(1)] \;\to\; 1/e \;\neq\; 0\,.$$

Corollary 9 applies when the larger population, n_A, has vote probability $p_A \geq \epsilon$. What if instead the smaller population n_B has $p_B \geq \epsilon$? If we assume that both $n_A \to \infty$ and $n_B \to \infty$, then we can strengthen Corollary 9 to assume that just $\max(p_A, p_B) \geq \epsilon$, i.e. that either one of them is bounded away from 0:

Corollary 10 *In the setup of Corollary 6, for any $\epsilon, r \in (0,1)$,*

$$\lim_{n_B \to \infty} \sup_{n_A \geq n_B/(1-r)} \sup_{\substack{p_A, p_B \in [0,1] \\ \max(p_A,p_B) \geq \epsilon}} \mathbf{P}(|X-Y| \leq \alpha) \;=\; 0\,.$$

Proof. Corollary 9 covers the case where $p_A \geq \epsilon$, so we assume here that $p_B \geq \epsilon$. Assume without loss of generality (by reducing ϵ if necessary) that $\epsilon \leq 1/2$. As before,

$$\mathbf{P}(|X-Y| \leq \alpha) \;=\; \sum_z \sum_{|d| \leq \alpha} \mathbf{P}[X=z+d]\;\mathbf{P}[Y=z]\,. \qquad (*)$$

Now, if $z \leq \epsilon n_B$, then since $p_B \geq \epsilon$ and $\epsilon \leq 1-\epsilon$,

$$\mathbf{P}[Y=z] \;=\; f(n_B, p_B; z) \;\leq\; \sup_{\epsilon \leq p \leq 1} \max_{0 \leq k \leq (1-\epsilon)n_B} f(n_B, p; k)\;[1+o(1)]\,.$$

Hence, by Proposition 8 (with $r = \epsilon$),

$$\mathbf{P}[Y = z] \;\leq\; \sqrt{1/2\pi n_B \epsilon(1-\epsilon)}\;[1 + o(1)] \;\to\; 0\,.$$

If instead $z \in (\epsilon n_B,\, n_B]$, then since $z \to \infty$ and $n_A - z \geq n_A - n_B \geq r n_B \to \infty$, using that $1 - (z/n_A) \geq 1 - (n_B/n_A) \geq 1 - (1-r) = r$, it follows from Lemmas 3 and 4(a) that

$$\mathbf{P}[X = z + d] \;\leq\; \sqrt{1/2\pi z[1 - (z/n_A)]}\;[1 + o(1)] \;\leq\; \sqrt{1/2\pi \epsilon n_B r}\;[1 + o(1)] \;\to\; 0\,.$$

Combining these two bounds, we compute from $(*)$ that

$$\begin{aligned}\mathbf{P}(|X - Y| \leq \alpha) \;\leq\; & \sum_{z \leq \epsilon n_B} \sum_{|d| \leq \alpha} \mathbf{P}[X = z + d]\,\sqrt{1/2\pi n_B \epsilon(1-\epsilon)}\;[1 + o(1)] \\ & + \sum_{z \in (\epsilon n_B,\, n_B]} \sum_{|d| \leq \alpha} \sqrt{1/2\pi \epsilon n_B r}\;\mathbf{P}[Y = z]\;[1 + o(1)] \\ \leq\; & (2\alpha + 1)\sqrt{1/2\pi n_B \epsilon(1-\epsilon)}\;[1 + o(1)] \;+\; (2\alpha + 1)\;\sqrt{1/2\pi \epsilon n_B r}\;[1 + o(1)]\,,\end{aligned}$$

which converges to 0 as $n_B \to \infty$, as claimed. ∎

Finally, considering the converse of Corollary 10 gives:

Corollary 11 *In the setup of Corollary 6, if $n_B \to \infty$ and $n_A \geq n_B/(1-r)$ for some $r \in (0,1)$, and* $\liminf \mathbf{P}(|X - Y| \leq \alpha) > 0$*, then we must have both $p_A \to 0$ and $p_B \to 0$, i.e. everyone's probability of voting must converge to zero as $n_A, n_B \to \infty$.*

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