When Can Martingales Avoid Ruin?

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Note: After completing this paper, it was discovered that similar results had been obtained previously [B. Davis, "Divergence properties of some martingale transforms", Annals of Mathematical Statistics **40** (1969), 1852–1854]. Thus, this paper was withdrawn at that point.

Abstract

We provide conditions that guarantee for a discrete time martingale M_n, either $\lim_{n\to\infty} M_n$ exists, or both $\limsup_{n\to\infty} M_n = \infty$ and $\liminf_{n\to\infty} M_n = -\infty$, a.s. on sample paths. A sufficient condition on the martingale is a uniform control on the ratio of the L^2 and L^1 norms of the increments. Near-symmetry of the increments provides an alternative condition. We also discuss a counterexample when these conditions are violated.

1. Introduction and Main Results

Martingales are the mathematical idealization of the idea of a "fair game". The martingale property says that for each play of the game, the expected change in one's fortune is zero. However, such a local definition of fairness does not automatically lead to a globally fair game. It is easy to construct pathological, unfair martingales M_n for which $\lim_{n\to\infty} M_n = \infty$ a.s. or for which $\lim_{n\to\infty} M_n = -\infty$ a.s.

In light of such examples, it is reasonable to ask under which additional assumptions either $\lim_{n\to\infty} M_n$ exists (as a finite limit), or both $\limsup_{n\to\infty} M_n = \infty$ and $\liminf_{n\to\infty} M_n = -\infty$, a.s. The latter property is equivalent to the sample path hitting all half-lines $(-\infty, a]$ and $[a, \infty)$, $a \in \mathbf{R}$, eventually (and hence infinitely often), and so we refer to such sample paths as *half-line recurrent* (HLR). We can think of this behavior as corresponding to a general form of ruin.

By the Chung-Fuchs theorem, any nondegenerate mean 0 random walk is recurrent. (See, for example, Section 3.2 of [1].) Consequently, almost all sample paths of such a random

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walk are HLR. It is easy to see that for continuous time martingales M_t with continuous sample paths, each path either converges or is HLR a.s. For this, let τ_a denote the hitting time of $(-\infty, a]$ and note that $M_{t\wedge\tau_a}$ is a martingale bounded below by a. By the martingale convergence theorem, $M_{t\wedge\tau_a}$ converges a.s. as $t \to \infty$. If the limit is M_{τ_a}, then M_t hits $(-\infty, a]$; otherwise, M_t converges. The argument for $[a, \infty)$ is the same. Similar reasoning also shows that for a discrete time martingale with increments X_i that are uniformly bounded below, almost all paths either converge or are HLR.

A simple example of a discrete time martingale where this property fails is obtained by taking $X_1, X_2, \ldots$ to be independent with

$$P\left(X_i = \frac{2^i}{2^i - 1}\right) = \frac{2^i - 1}{2^i} \quad \text{and} \quad P(X_i = -2^i) = \frac{1}{2^i}\,. \tag{1.1}$$

Since $\sum_{i=1}^{\infty} P(X_i = -2^i) < \infty$, it follows from the Borel-Cantelli lemma that $X_i > 1$ for all but finitely many i. Hence, $M_n = X_1 + \cdots + X_n \to \infty$ a.s. as $n \to \infty$.

Such counterexamples can be avoided by controlling the tails of the increments X_i. Conditions for this are given by our two main results, Theorems 1.1 and 1.4. Theorem 1.1 assumes an upper bound on the conditional variances and Theorem 1.4 makes a symmetry assumption.

Throughout the paper,

$$M_n = x_0 + X_1 + \cdots + X_n, \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

will be a martingale or supermartingale with respect to a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$ of σ-fields on a probability space $(\Omega, \mathcal{F}, P)$. The martingale (supermartingale) property says that $E[X_i \mid \mathcal{F}_{i-1}] = 0$ (≤ 0) a.s. for all $i \geq 1$. The conditional variances will be denoted by

$$\sigma_i^2 = E[X_i^2 | \mathcal{F}_{i-1}]\,. \tag{1.3}$$

Theorem 1.1. *Suppose that M_n is a martingale with increments X_n satisfying*

$$\sigma_n^2 \,\leq\, K\, \Big((E[|X_n| \,|\, \mathcal{F}_{n-1}])^2 \,\vee\, E[X_n^2;\; |X_n| \leq A \,|\, \mathcal{F}_{n-1}] \Big) \tag{1.4}$$

a.s. for all $n = 1, 2, \ldots$ and fixed $A, K \in [1, \infty)$. Then,

$$P(\text{either } M_n \text{ converges or } M_n \text{ is HLR}) = 1\,. \tag{1.5}$$

The first bound on the right side of (1.4) is the more important one and is used for the following two corollaries. We will demonstrate Theorem 1.1 in Section 2. There, we will show that M_n converges when $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, and that M_n is otherwise HLR. In order to show that M_n is HLR, it will be enough to show $\tau \stackrel{\text{def}}{=} \tau_0 < \infty$ must hold, by translating and reflecting M_n.

A *martingale transform of a random walk* (MTRW) is a martingale with $X_i = \theta_{i-1} Z_i$, where $Z_1, Z_2, \dots$ are independent and identically distributed, $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$, and $\theta_i \in \mathcal{F}_i$. It is the discrete analogue of a stochastic integral. A special case of Theorem 1.1 is the following result. (We denote by Z a random variable with the common distribution of Z_i.)

Corollary 1.2. *A MTRW with $E[Z^2] < \infty$ satisfies (1.5).*

Here, we can set $K = E[Z^2] \,/\, E[|Z|]^2$ and employ the first bound on the right side of (1.4). In Section 4, we will give an example of a MTRW with $E[|Z|^p] < \infty$, for given $p \in [1, 2)$, but for which (1.5) does not hold.

Another family of martingales is given by *finitely inhomogeneous random walks* (FIRW's). Let $Z_i^{(j)}$, $i = 1, 2, \dots$ and $j = 1, \dots, J$ denote independent mean 0 random variables with distribution functions $F^{(j)}$, and set $\mathcal{F}_n = \sigma(Z_i^{(j)}, i = 1, \dots, n, j = 1, \dots, J)$. Also, let $\theta_i \in \mathcal{F}_i$, where θ_i takes values $1, 2, \dots, J$. Then, $M_n = x_0 + Z_1^{(\theta_0)} + \dots + Z_n^{(\theta_{n-1})}$ is a FIRW.

Corollary 1.3. *A FIRW, with $E[(Z^{(j)})^2] < \infty$ for all j, satisfies (1.5).*

Corollary 1.3 was shown in [2] (see also [4]). There, examples are given of FIRW's with $E[|Z^{(j)}|^p] < \infty$, for given $p \in [1, 2)$, but for which (1.5) does not hold.

When (1.4) does not hold, appropriate symmetry conditions on X_i are still enough to imply (1.5). Here, $X^+ = X \vee 0$ and $X^- = (-X) \vee 0$.

Theorem 1.4. *Suppose that M_n is a supermartingale with increments X_n satisfying*

$$E[X_n^+;\ X_n^+ > x \,|\, \mathcal{F}_{n-1}] \ \geq \ E[X_n^-;\ X_n^- > bx \,|\, \mathcal{F}_{n-1}] \tag{1.6}$$

a.s. for all $n = 1, 2, \dots$ and $x \geq x_1$, for fixed b and $x_1 > 0$. Then,

$$P(\text{either } M_n \text{ converges or } \liminf_{n\to\infty} M_n = -\infty) = 1\,. \tag{1.7}$$

If M_n is a martingale satisfying both (1.6) and its analogue with the roles of X_n^+ and X_n^- reversed, then (1.5) holds.

We will demonstrate Theorem 1.4 in Section 3. The proof is relatively simple and uses the Skorokhod embedding.

This paper was originally motivated by a question arising from the paper [3] about whether a MTRW, for which Z_i are standard Gaussian random variables, must satisfy (1.5). (The answer is yes, on account of both Corollary 1.2 and Theorem 1.4.)

2. Proof of Theorem 1.1

Before proving Theorem 1.1, we first present three lemmas. The first lemma exploits (1.4) to obtain a lower bound on the L^1 norm of a stopped martingale with bounded increments. Recall that σ_n^2 is defined in (1.3).

Lemma 2.1. *Let $M_n = x_0 + X_1 + \cdots + X_n$ be a martingale satisfying (1.4). Also, assume that for some fixed $A > 0$, $\sigma_n \le A$ always holds, and set*

$$\alpha = \min\{n : |M_n| \ge A\}. \tag{2.1}$$

Then, for any stopping time γ,

$$E\Big[|M_{\alpha\wedge\gamma}|\Big] \ge E\left[\sum_{n=1}^{\alpha\wedge\gamma} \sigma_n^2\right] / \, 64AK^2 \, . \tag{2.2}$$

Proof. For a given n, we first consider realizations on which the first bound on the right side of (1.4) holds. We note that the general inequality,

$$P(|X| \ge c) \ge (E[|X|] - c)^2 \, / \, E[X^2] \qquad \text{for} \quad c \le E[|X|] \, ,$$

follows from Schwarz's inequality $E[|XY|]^2 \le E[X^2]E[Y^2]$ with $Y = 1_{|X|\ge c}$. Applying this to $X = |X_n|$ with $c = \sigma_n/2\sqrt{K}$, and using the first half of (1.4), we have

$$P(|X_n| \ge \sigma_n \, / \, 2\sqrt{K} \, | \, \mathcal{F}_{n-1}) \ge 1/4K \, . \tag{2.3}$$

Let

$$\varphi(x) \;=\; \begin{cases} x^2 & \text{for } |x| \le 2A \\ 4A(|x| - 2A) + 4A^2 & \text{for } |x| > 2A \, . \end{cases}$$

Note that φ is convex, with $\varphi''(x) = 2$ for $|x| < 2A$. So, for $|x_0| \vee c \le A$ and $|x| \ge c$, it follows that

$$\varphi(x_0 + x) \ge \varphi(x_0) + x\varphi'(x_0) + c^2 \, .$$

Therefore, if X is any random variable with $E[X] = 0$,

$$E[\varphi(x_0 + X)] \ge \varphi(x_0) + P(|X| \ge c) \; c^2 \tag{2.4}$$

for $|x_0| \vee c \le A$. Applying (2.4) to $x_0 = M_{n-1}$, $X = X_n$ and $c = \sigma_n/2\sqrt{K} \le A$, we therefore have

$$E[\varphi(M_n) \, | \, \mathcal{F}_{n-1}] \ge \varphi(M_{n-1}) + P\Big(|X_n| \ge \sigma_n/2\sqrt{K} \, | \, \mathcal{F}_{n-1}\Big) \, \sigma_n^2/4K \, , \tag{2.5}$$

for $|M_{n-1}| \le A$. It follows from (2.3) and (2.5), that for $|M_{n-1}| \le A$,

$$E[\varphi(M_n) - \varphi(M_{n-1}) \, | \, \mathcal{F}_{n-1}] \ge \sigma_n^2 \, / \, 16K^2 \, . \tag{2.6}$$

Next consider realizations on which the second bound on the right side of (1.4) holds for given n. For $|x_0| \leq A$,

$$\varphi(x_0 + x) \geq \begin{cases} \varphi(x_0) + x\varphi'(x_0) + x^2 & \text{for } |x| \leq A \\ \varphi(x_0) + x\varphi'(x_0) & \text{for } |x| > A\,. \end{cases}$$

Therefore, if X is any random variable with $E[X] = 0$,

$$E[\varphi(x_0 + X)] \geq \varphi(x_0) + E[X^2;\ |X| \leq A]\,.$$

Setting $x_0 = M_{n-1}$ and $X = X_n$, and then applying the second half of (1.4), implies that

$$E[\varphi(M_n) - \varphi(M_{n-1}) \,|\, \mathcal{F}_{n-1}] \ \geq\ E[X_n^2;\ |X_n| \leq A \,|\, \mathcal{F}_{n-1}] \ \geq\ \sigma_n^2/K\,.$$

Hence, (2.6) holds in this setting as well.

We note that $\varphi(M_0) \geq 0$ and $\{\alpha \wedge \gamma < n\} \in \mathcal{F}_{n-1}$. So, iterating the quantity in (2.6) implies that

$$E[\varphi(M_{\alpha\wedge\gamma})] \ \geq\ E\left[\sum_{n=1}^{\alpha\wedge\gamma} \sigma_n^2\right] \,/\, 16K^2\,.$$

Since $\varphi(x) \leq 4A|x|$ for all x, (2.2) follows. ∎

As noted in the introduction, when the increments of a martingale are bounded from below, (1.5) always holds. The next lemma gives an upper bound on the probability of large negative jumps, in terms of the conditional variances in (1.3).

Lemma 2.2. *Let $M_n = x_0 + X_1 + \cdots + X_n$ be a martingale. For any fixed $B > 0$, let*

$$\beta = \min\{n : X_n^- \geq B\}. \tag{2.7}$$

Then, for any stopping time γ,

$$P(\beta \leq \gamma) \leq E\left[\sum_{n=1}^{\gamma} \sigma_n^2\right] /B^2\,. \tag{2.8}$$

Proof. Since $\{\gamma \geq n\} \in \mathcal{F}_{n-1}$,

$$P(\beta \leq \gamma) = E\left[\sum_{n=1}^{\infty} \mathbf{1}_{\gamma\geq n}\mathbf{1}_{\beta=n}\right] = E\left[\sum_{n=1}^{\gamma} E[\mathbf{1}_{\beta=n} \,|\, \mathcal{F}_{n-1}]\right]\,.$$

This is

$$\leq E\left[\sum_{n=1}^{\gamma} E[\mathbf{1}_{X_n^-\geq B} \,|\, \mathcal{F}_{n-1}]\right] \leq E\left[\sum_{n=1}^{\gamma} \sigma_n^2\right] \,/\, B^2\,,$$

with the last inequality following from the conditional Chebyshev inequality. ∎

In Lemma 2.3, we employ Lemmas 2.1 and 2.2 to show there is a positive probability of hitting $(-\infty, 0]$ when $\sum_{i=1}^{n} \sigma_i^2$ is sufficiently large. Let Δ denote the set on which $\sum_{i=1}^{\infty} \sigma_i^2 = \infty$, with $d_0 = P(\Delta)$. We let τ denote the hitting time of $(-\infty, 0]$, and choose n_0 to be the first time at which

$$P\left(\sum_{i=1}^{n_0} \sigma_i^2 \geq N\right) \geq 3d_0/4 \tag{2.9}$$

with a given $N < \infty$.

Lemma 2.3. *Let $M_n = x_0 + X_1 + \cdots + X_n$, $x_0 > 0$, be a martingale satisfying (1.4), and assume that $d_0 > 0$. Then,*

$$P(\tau \leq n_0) \geq C_1 d_0^2 / K^4 \tag{2.10}$$

for $N = C_2 K^4 x_0^2 / d_0^2$ and appropriate constants $C_1 > 0$ and C_2.

Proof. Let $\nu = \min\{n : \sum_{i=1}^{n} \sigma_i^2 \geq N\}$. We use ν to truncate M_n by setting $\overline{M}_n = M_n$ for $n < \nu$ and

$$\overline{M}_n = M_{\nu-1} + X_\nu \sigma_\nu^{-1} \left(N - \sum_{i=1}^{\nu-1} \sigma_i^2\right)^{1/2} \qquad \text{for } n \geq \nu\,. \tag{2.11}$$

$\overline{M}_n$ is a martingale starting at x_0 whose increments satisfy (1.4). The ν^{th} increment of M_n has been defined so that $\sum_{i=1}^{\nu} \overline{\sigma}_i^2 = N$ on $\nu < \infty$. Clearly, $\overline{\tau} < \nu$ exactly when $\tau < \nu$, and $\overline{\tau} > \nu$ cannot occur. (The terms $\overline{\sigma}_i^2$ and $\overline{\tau}$ are the analogues of σ_i^2 and τ.) Moreover, since the product of the two terms multiplying X_ν in (2.11) is at most 1, $\overline{\tau} = \nu < \infty$ can only occur when $\tau = \nu < \infty$ does. So, by comparing M_n with $\overline{M}_n$, it suffices to demonstrate (2.10) under the additional assumption that on $\nu < \infty$,

$$\sum_{i=1}^{n} \sigma_i^2 = N \qquad \text{for } n \geq \nu\,. \tag{2.12}$$

We set

$$\mu = \alpha \wedge \beta \wedge \tau \wedge n_0\,,$$

where α is given by (2.1) and β is given by (2.7); A and B are fixed and will later be chosen appropriately. We proceed to obtain upper bounds on $E[|M_\mu|]$. We consider two cases, depending on whether or not

$$P(\alpha \leq n_0) \leq \frac{1}{6}\, d_0\,. \tag{2.13}$$

Assume (2.13) holds. By choosing B large enough so that $N/B^2 \leq \frac{1}{24}d_0$, it follows from Lemma 2.2 and (2.12) that

$$P(\beta \leq n_0) \leq \frac{1}{24}\, d_0\,. \tag{2.14}$$

We can assume that $P(\tau \leq n_0) \leq \frac{1}{24}d_0$. (Otherwise, (2.10) is satisfied with $C_1 = \frac{1}{24}$, since $K \geq 1$ is assumed.) Also, by (2.9), $P(n_0 < \nu) \leq 1 - \frac{3}{4}d_0$. So,

$$P(\mu < \nu) \leq P(\alpha \wedge \beta \wedge \tau \leq n_0) + P(n_0 < \nu) \leq 1 - \frac{1}{2}d_0\,. \tag{2.15}$$

We will later choose A and N so that $N \leq A^2$. It follows from this and (2.12) that $\sigma_i \leq A$ always holds, and so we may employ Lemma 2.1. Together with (2.12) and (2.15), the lemma implies that

$$E[|M_\mu|] \;\geq\; E\left[\sum_{i=1}^{\nu} \sigma_i^2;\; \mu \geq \nu\right] \Big/ \, 64AK^2 \;\geq\; Nd_0/128AK^2\,. \tag{2.16}$$

If, on the other hand, (2.13) does not hold, then

$$P(\alpha \leq \beta \wedge \tau \wedge n_0) \geq P(\alpha \leq n_0) - P(\beta \leq n_0) - P(\tau \leq n_0) \geq \frac{1}{12}\, d_0\,,$$

where the last inequality follows from (2.14) and our assumption on τ. Since $|M_\alpha| \geq A$, it follows that

$$E[|M_\mu|] \geq \frac{1}{12} A d_0\,. \tag{2.17}$$

By (2.16) and (2.17),

$$E[|M_\mu|] \geq \min\left\{\frac{1}{128}\,\frac{Nd_0}{AK^2},\; \frac{1}{12}Ad_0\right\}$$

always holds. Since M_n is a martingale and μ is bounded,

$$E[|M_\mu|] = E[M_\mu^+] + E[M_\mu^-] = 2\,E[M_\mu^-] + x_0$$

by the optional sampling theorem. So,

$$E[M_\mu^-] \geq \min\left\{\frac{1}{256}\,\frac{Nd_0}{AK^2},\; \frac{1}{24}Ad_0\right\} - \frac{1}{2}\,x_0\,. \tag{2.18}$$

On the set where $\beta = \mu$, we have $\beta \leq n_0$ and $M_\mu^- \leq X_\beta^-$, and therefore,

$$E[M_\mu^-;\; \beta = \mu] \;\leq\; \sum_{i=1}^{n_0} E[X_i^-;\; X_i^- \geq B] \;=\; \sum_{i=1}^{n_0} E\Big[E[X_i^-;\; X_i^- \geq B \,|\, \mathcal{F}_{n-1}]\Big]$$

$$\leq \frac{1}{B}\sum_{i=1}^{n_0} E[\sigma_i^2] \leq N/B\,,$$

where the last inequality follows from (2.12). It follows from this and (2.18), that

$$E[M_\mu^-;\ \beta > \mu] \ge \min\left\{\frac{1}{256}\frac{Nd_0}{AK^2},\ \frac{1}{24}Ad_0\right\} - \frac{1}{2}x_0 - \frac{N}{B}\,. \tag{2.19}$$

When $M_\mu^- > 0$, then $\tau \le n_0$ must hold, and when $\beta > \mu$, then $M_\mu^- < B$. So,

$$P(\tau \le n_0) \ge P(M_\mu^- > 0) \ge P(M_\mu^- > 0;\ \beta > \mu) \ge E[M_\mu^-;\ \beta > \mu]\,/\,B\,.$$

From this and (2.19), it follows that

$$P(\tau \le n_0) \ge \min\left\{\frac{1}{256}\frac{Nd_0}{ABK^2},\ \frac{1}{24}\frac{Ad_0}{B}\right\} - \frac{1}{2}\frac{x_0}{B} - \frac{N}{B^2}\,.$$

The choice of

$$A = 2^9K^2x_0/d_0, \quad B = 2^{19}K^4x_0/d_0^2, \quad N = 2^{18}K^4x_0^2/d_0^2$$

produces the bound

$$P(\tau \le n_0) \ge d_0^2\,/\,2^{19}K^4\,,$$

which implies (2.10). ∎

We now prove Theorem 1.1.

Proof of Theorem 1.1. It is not difficult to see that M_n converges on Δ^c. For a given N, define ν as at the beginning of the proof of Lemma 2.3. We set $\overline{M}_n = M_{n\wedge(\nu-1)}$, which is also a martingale starting at x_0. Since $\sum_{i=1}^{\infty}\overline{\sigma}_i^2 \le N$ (where $\overline{\sigma}_i^2$ is the analogue of σ_i^2), it follows that

$$E\Big[|\overline{M}_n - x_0|\Big]^2 \le E\Big[(\overline{M}_n - x_0)^2\Big] \le N \qquad \text{for all } n\,.$$

So, by the martingale convergence theorem, $\overline{M}_n$ converges a.s. to a finite limit. On Δ^c, $M_n = \overline{M}_n$ for all n, for large enough N (where N depends on the realization). It follows that M_n converges a.s. to a finite limit on Δ^c.

We still need to show that M_n is half-line recurrent on Δ. To do so, it is enough to show $\tau < \infty$ a.s. on Δ for any $x_0 > 0$. The basic idea is to repeatedly apply Lemma 2.3 to martingales $M_n^{(j)}$, $j = 1, 2, \ldots$, where $M_n^{(j)} = M_{m_j+n}$ for appropriately increasing m_j.

We begin by introducing the following quantities. Choose N_j, $j = 1, 2, \ldots$, to be an increasing sequence, where N_j is large enough so that

$$P\left(N_j < \sum_{i=1}^{\infty}\sigma_i^2 < \infty\right) \le b_j\,, \tag{2.20}$$

with $b_j = C_1 \,/\, K^4 2^{j+2}$ and C_1 is as in (2.10). Choose m_j to be an increasing sequence where m_j is large enough so that

$$P\left(\sum_{i=1}^{m_j} \sigma_i^2 \le N_j < \sum_{i=1}^{\infty} \sigma_i^2\right) \le b_j \,. \tag{2.21}$$

The m_j will be the starting times for the martingales $M_n^{(j)}$ referred to above. Also, choose y_j large enough so that

$$P(M_{m_j} > y_j) \le b_j \,. \tag{2.22}$$

We will later choose m_j to depend on y_{j-1} in an appropriate manner.

We also introduce the following sets. Let

$$F_j = \Delta \cap \{\tau > m_j\}\,, \quad G_j = \left\{\sum_{i=1}^{m_j} \sigma_i^2 > N_j\right\}\,, \quad H_j = G_j \cap \{\tau > m_j\} \cap \{M_{m_j} \le y_j\}\,.$$

By (2.20),

$$P(H_j \cap \Delta^c) \le P(F_j^c \cap H_j) \le b_j \,, \tag{2.23}$$

and by (2.21) and (2.22),

$$P(F_j \cap H_j^c) \le 2\, b_j \,. \tag{2.24}$$

We will show $P(F_j)$ decreases geometrically rapidly as $j \to \infty$; from this, it will follow that $\tau < \infty$ occurs a.s. on Δ. Most of the following estimates will involve $P(H_j)$ rather than $P(F_j)$; (2.23) and (2.24) will be applied to $P(H_j)$ to bound $P(F_j)$.

Let $M_n^{(j)} = M_{m_j+n}$ denote the martingale with σ-fields $\mathcal{F}_n^{(j)} = \mathcal{F}_{m_j+n}$, and $\tau^{(j)}$ the first hitting time of $(-\infty, 0]$ for $M_n^{(j)}$. Define $n_0^{(j)}$ analogously to n_0 in (2.9), with $N = C_2 K^4 y_j^2 \,/\, d_0^2$. Since (1.4) is still satisfied and $M_0^{(j)} \le y_j$ on H_j, it follows from Lemma 2.3 that on H_j,

$$P(\tau^{(j)} \le n_0^{(j)} \,|\, \mathcal{F}_{m_j}) \;\ge\; a\, P(\Delta \,|\, \mathcal{F}_{m_j})^2$$

a.s. for $a = C_1 \,/\, K^4$; we assume WLOG that $a \le 1/2$. It therefore follows from Jensen's inequality, $H_j \in \mathcal{F}_{m_j}$, and (2.23), that

$$P(H_j;\; \tau^{(j)} \le n_0^{(j)}) \;\ge\; a \int_{H_j} P(\Delta \,|\, \mathcal{F}_{m_j})^2 dP \;\ge\; a\, P(H_j \cap \Delta)^2 \,/\, P(H_j) \;\ge\; a(P(H_j) - 2b_j)\,.$$

Consequently,

$$P(H_j;\; \tau^{(j)} > n_0^{(j)}) \le (1-a)\, P(H_j) + b_j \,. \tag{2.25}$$

Choose m_{j+1} large enough so that both $m_{j+1} \ge m_j + n_0^{(j)}$ and (2.21) hold. Then,

$$P(F_{j+1}) = P(F_j;\; \tau > m_{j+1}) \;\le\; P(H_j;\; \tau > m_{j+1}) + P(F_j \cap H_j^c)$$

$$\leq \; (1-a)\,P(H_j) + 3\,b_j \;\leq\; (1-a)\,P(F_j) + 4\,b_j\,, \tag{2.26}$$

where the middle inequality follows from (2.24) and (2.25), and the last inequality follows from (2.23). If one assumes $P(F_j) \leq (1-a/2)^{j-1}$, then it follows from (2.26) that $P(F_{j+1}) \leq (1-a/2)^j$. By induction, the last inequality therefore holds for all j. Letting $j \to \infty$, this implies that $\tau < \infty$ occurs a.s. on Δ, which completes the proof of Theorem 1.1. ∎

3. Proof of Theorem 1.4

The proof of Theorem 1.4 employs a version of the well known Skorokhod embedding, which we first review. The embedding is most often applied to the sums $M_n = X_1 + \ldots + X_n$ of a sequence $X_1, X_2, \ldots$ of i.i.d. random variables with mean 0. One can embed M_n in a probability space $(\Omega, \mathcal{G}, P)$ supporting a standard Brownian motion $W(t)$ so that at appropriate stopping times $\alpha_0 = 0, \alpha_1, \alpha_2, \ldots$, the differences $W(\alpha_i) - W(\alpha_{i-1})$ are i.i.d. having the same distributions as X_1. Setting $\widetilde{M}_n = W(\alpha_n)$, the sequence $\widetilde{M}_n$ therefore has the same joint distributions as M_n. The stopping time α_i is defined as the first time after α_{i-1} at which $W(t)$ hits either $W(\alpha_{i-1}) + Y_i$ or $W(\alpha_{i-1}) - Z_i$, where $Y_i, Z_i \geq 0$ are appropriately chosen random variables that are independent of $W(t)$. The choice of Y_i and Z_i can be made in various ways. (See [5] for a detailed survey of the Skorokhod embedding.)

Here, we use the original embedding by Skorokhod in [6], where one chooses Y_i and Z_i so that $Y_i(\omega_1) < Y_i(\omega_2)$ implies $Z_i(\omega_1) \leq Z_i(\omega_2)$ for $\omega_\ell \in \Omega$. For this choice, $Y_i = x^+$ implies that $Z_i \leq x^-$, where x^- is the smallest value for which

$$E[X_i^+;\; X_i^+ > x^+] \;\geq\; E[X_i^-;\; X_i^- > x^-]\,. \tag{3.1}$$

(When the distribution of X_i is continuous, $Z_i = x^-$, and the outside inequality in (3.1) is replaced by equality. In [6], $E[X_i^2] < \infty$ is assumed, although this is not necessary, as pointed out in [5] and elsewhere.)

The same embedding still applies when $X_1, X_2, \ldots$ are replaced by the increments of the martingale $M_n = x_0 + X_1 + \ldots + X_n$. Stopping times $\alpha_0 = 0, \alpha_1, \alpha_2, \ldots$ can be chosen so that $\widetilde{X}_i \stackrel{\text{def}}{=} W(\alpha_i) - W(\alpha_{i-1})$ has the same joint distributions as X_i, and so $\widetilde{M}_n \stackrel{\text{def}}{=} W(\alpha_n)$ has the same joint distributions as M_n. As before, Y_i and Z_i are chosen so $Y_i(\omega_1) < Y_i(\omega_2)$ implies $Z_i(\omega_1) \leq Z_i(\omega_2)$. In this setting, $Y_i = x^+$ implies $Z_i \leq x^-$, where x^- is the smallest value for which

$$E[\widetilde{X}_i^+;\; \widetilde{X}_i^+ > x^+ \,|\, \widetilde{\mathcal{F}}_{i-1}] \;\geq\; E[\widetilde{X}_i^-;\; \widetilde{X}_i^- > x^- \,|\, \widetilde{\mathcal{F}}_{i-1}]\,, \tag{3.2a}$$

where $\widetilde{\mathcal{F}}_n = \sigma(\widetilde{X}_1, \dots, \widetilde{X}_n)$. This is equivalent to

$$E[X_i^+;\ X_i^+ > x^+ \,|\, \mathcal{F}'_{i-1}] \ \geq\ E[X_i^-;\ X_i^- > x^- \,|\, \mathcal{F}'_{i-1}]\,, \tag{3.2b}$$

where $\mathcal{F}'_n = \sigma(X_1, \dots, X_n) \subset \mathcal{F}_n$. (If one wishes, one can WLOG set $\mathcal{F}_n = \mathcal{F}'_n$, when given M_n.) Let N_t denote the largest n with $\alpha_n \leq t$, and $\mathcal{G}_t$ the σ-field generated by $W(s)$, $s \leq t$, and Y_i and Z_i, $i \leq N_t + 1$. The random variables Y_i, Z_i, and $W(t)$ can be chosen so the increments of $W(s)$, on $s > t$, are independent of $\mathcal{G}_t$. (This requirement, together with (3.2) and the property that $W(\alpha_n)$ and M_n have the same joint distribution, completely specifies the joint distribution of $W(t)$, Y_i, and Z_i.) It follows that $W(t)$ will be a martingale with respect to $\mathcal{G}_t$.

Proof of Theorem 1.4. By the Doob decomposition, a supermartingale M_n can be written as $M_n = M'_n - A_n$, where M'_n is a martingale and A_n is a nondecreasing sequence. So, it suffices to show (1.7) under the assumption that M_n is a martingale. The second statement in the theorem is an immediate consequence of (1.7) applied to both M_n and $-M_n$.

In order to show (1.7) for the martingale M_n, we employ the above Skorokhod embedding, with the random variables introduced there. Let Δ denote the set of realizations where $\alpha_\infty \stackrel{\text{def}}{=} \lim_{n\to\infty} \alpha_n = \infty$. It is easy to see that on Δ^c,

$$\lim_{n\to\infty} \widetilde{M}_n = \lim_{n\to\infty} W(\alpha_n) = W(\alpha_\infty)\,.$$

So, $\widetilde{M}_n$ converges on Δ^c.

We will show that $\liminf_{n\to\infty} \widetilde{M}_n = -\infty$ a.s. on Δ. To do so, it is enough to show $\widetilde{\tau} < \infty$ a.s. on Δ for any x_0, where $\widetilde{\tau}$ is the hitting time of $(-\infty, 0]$ by $\widetilde{M}_n$. Let $\beta < \infty$ be the stopping time at which $W(t)$ first hits $-bx_1$. By the optional sampling theorem,

$$P\left(W(\alpha_{N_\beta+1}) = W(\alpha_{N_\beta}) - Z_{N_\beta+1} \,|\, \mathcal{G}_\beta\right) \ =\ \frac{W(\alpha_{N_\beta}) + Y_{N_\beta+1} + bx_1}{Y_{N_\beta+1} + Z_{N_\beta+1}} \tag{3.3}$$

a.s. on $\beta < \alpha_\infty$. Here, we are using that $\alpha_{N_\beta+1} > \beta$, but $\alpha_{N_\beta} \leq \beta$ and $Y_{N_\beta+1}, Z_{N_\beta+1} \in \mathcal{G}_\beta$, that $W(\beta) = -bx_1$, and that $W(\alpha_{N_\beta+1})$ takes only the values $W(\alpha_{N_\beta}) + Y_{N_\beta+1}$ and $W(\alpha_{N_\beta}) - Z_{N_\beta+1}$. Since $W(\alpha_{N_\beta}) \geq -bx_1$, the right side of (3.3) is at least $Y_{N_\beta+1} \,/\, (Y_{N_\beta+1} + Z_{N_\beta+1})$. Since $\widetilde{M}_n = W(\alpha_n)$, it therefore follows that

$$P\left(\widetilde{M}_{N_\beta+1} = \widetilde{M}_{N_\beta} - Z_{N_\beta+1} \,|\, \mathcal{G}_\beta\right) \ \geq\ \frac{Y_{N_\beta+1}}{Y_{N_\beta+1} + Z_{N_\beta+1}} \tag{3.4}$$

a.s. on $\beta < \alpha_\infty$.

On the other hand, it follows from (1.6), and our choice of x^- as the smallest value at which (3.2) holds, that

$$x^- \leq b(x^+ \vee x_1)\,. \tag{3.5}$$

So, on $Y_{N_\beta+1} \geq x_1$, the right side of (3.4) is at least $1/(b+1)$. Since

$$\widetilde{M}_{N_\beta} - Z_{N_\beta+1} = W(\alpha_{N_\beta}) - Z_{N_\beta+1} < W(\beta) = -bx_1 < 0\,,$$

the event on the left side of (3.4) is contained in $\{\tilde{\tau} \leq N_\beta + 1\}$. Moreover, on $Y_{N_\beta+1} < x_1$, one has $Z_{N_\beta+1} < bx_1$, and so

$$\widetilde{M}_{N_\beta} = W(\alpha_{N_\beta}) \leq W(\beta) + Z_{N_\beta+1} < 0\,, \tag{3.6}$$

which implies $\tilde{\tau} \leq N_\beta$.

Together, these two cases imply that

$$P\,(\tilde{\tau} \leq N_\beta + 1\,|\,\mathcal{G}_\beta) \,\geq\, \frac{1}{b+1} \tag{3.7}$$

a.s. on $\beta < \alpha_\infty$. Once can continue by setting $\beta_1 = \beta$, and defining $\beta_2, \beta_3, \ldots$ inductively, with β_j being the first time after $\alpha_{N_{\beta_{j-1}}+1}$ when $W(t)$ hits $-bx_1$. The same reasoning as before produces the analogue of (3.7), but with β_j substituted for β. Consequently,

$$P\left(\tilde{\tau} > N_{\beta_j} + 1;\ \Delta\right) \,\leq\, P\left(\tilde{\tau} > N_{\beta_j} + 1;\ \beta_j < \alpha_\infty\right) \,\leq\, \left(\frac{b}{b+1}\right)^j .$$

Letting $j \to \infty$ implies that $\tilde{\tau} < \infty$ a.s. on Δ, as desired. ∎

4. A Martingale Transform Random Walk Example

By Corollary 1.2, the limit of a martingale transform of a random walk, with $E[Z^2] < \infty$, satisfies (1.5). We show here that (1.5) need not hold when $E[|Z|^p] < \infty$, $p \in [1,2)$, is instead assumed.

Fix $p \in [1,2)$. Let $a_j = 2^{2^j}$, $j = 2, 3, \ldots$, and let F be the distribution function with mass $2^{-j}a_j^{-1}$ at $-a_j^{1/p}$ for $j = 2, 3, \ldots$, and masses at 0 and 1 so that $\int_{-\infty}^{\infty} x\,F(dx) = 0$. Let Z_i be i.i.d. random variables with distribution F. We choose the MTRW $M_n = \theta_0 Z_1 + \cdots + \theta_{n-1} Z_n$ with $\theta_{n-1} = a_{\gamma_n}^{-1/2}$ where

$$\gamma_n = \max_{i \leq n-1} \lfloor M_i \rfloor \vee 2\,.$$

($\lfloor x \rfloor$ denotes the integer part of x.) It is easy to check that $E[|Z^p|] < \infty$, but $E[Z^2] = \infty$. We will show:

Proposition 4.1. *As $n \to \infty$, $M_n \to \infty$ a.s.*

Proof. Set $\overline{Z}_i = Z_i \mathbf{1}_{Z_i > -a_{\gamma_i}^{1/p}}$ and $\tau(j) = \min\{n : M_n \geq j\}$. Note that for $n > \tau(j)$, one has $\gamma_n \geq j$, and so $a_{\gamma_n} \geq a_j$. Our approach will be to compare M_n over the intervals $(\tau(j), \tau(j+1)]$, $j = 2, 3, \ldots$, with the random variables $\overline{M}_n^{(j)}$ obtained by replacing Z_i with $\overline{Z}_i$ there. We will show that the probability $\overline{Z}_i \neq Z_i$ is negligible for large j, but that $\overline{M}_n^{(j)}$ has a substantial cumulative positive drift over the interval. Comparison with M_n will imply that $M_n \to \infty$ as $n \to \infty$.

It is easy to see that

$$P(\overline{Z}_i \neq Z_i \,|\, \mathcal{F}_i) = \sum_{k=\gamma_i}^{\infty} 2^{-k} a_k^{-1} \leq C_1 2^{-\gamma_i} a_{\gamma_i}^{-1}, \tag{4.1}$$

for an appropriate constant C_1. Also,

$$E[\overline{Z}_i \,|\, \mathcal{F}_{i-1}] = E\left[-Z_i \mathbf{1}_{Z_i \leq -a_{\gamma_i}^{1/p}} \,|\, \mathcal{F}_{i-1}\right] = \sum_{k=\gamma_i}^{\infty} 2^{-k} a_k^{\frac{1}{p}-1} \geq C_2 2^{-\gamma_i} a_{\gamma_i}^{\frac{1}{p}-1},$$

for an appropriate constant $C_2 > 0$, and so

$$E[\theta_{i-1}\overline{Z}_i \,|\, \mathcal{F}_{i-1}] \geq C_2 2^{-\gamma_i} a_{\gamma_i}^{\frac{1}{p}-\frac{3}{2}}. \tag{4.2}$$

Similarly,

$$\mathrm{Var}(\theta_{i-1}\overline{Z}_i \,|\, \mathcal{F}_{i-1}) \leq E[\theta_{i-1}^2 \overline{Z}_i^2 \,|\, \mathcal{F}_{i-1}] = a_{\gamma_i}^{-1} \sum_{k=2}^{\gamma_i - 1} 2^{-k} a_k^{\frac{2}{p}-1} \leq C_3 2^{-\gamma_i} a_{\gamma_i}^{\frac{1}{2p}-\frac{5}{4}}, \tag{4.3}$$

for an appropriate constant C_3. The last inequality uses the rapid growth of a_k.

Set $\mu_j = C_2 2^{-j} a_j^{\frac{1}{p}-\frac{3}{2}}$. By (4.1) and our observation that $a_{\gamma_i} \geq a_j$ for $i > \tau(j)$,

$$P\Big(\overline{Z}_i \neq Z_i \text{ for some } i \in \big(\tau(j),\ \tau(j) + 2\lceil \mu_j^{-1} \rceil\big]\Big) \leq C_1 2^{-j+2} a_j^{-1} \mu_j^{-1} = C_4 a_j^{\frac{1}{2}-\frac{1}{p}}, \tag{4.4}$$

for $C_4 = 4C_1C_2^{-1}$. ($\lceil x \rceil$ denotes the smallest integer at least x.) Also, by the Kolmogorov inequality for martingales, (4.2), and (4.3),

$$P\left(\sum_{i=\tau(j)+1}^{n} (\theta_{i-1}\overline{Z}_i - \mu_j) \leq -1 \text{ for some } n \in \big(\tau(j),\ \tau(j) + 2\lceil \mu_j^{-1} \rceil\big]\right)$$

$$\leq 4\mu_j^{-1} \cdot C_3 2^{-j} a_j^{\frac{1}{2p}-\frac{5}{4}} = C_5\, a_j^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}$$

for $C_5 = 4C_2^{-1}C_3$. Together with (4.4), this implies that

$$P\left(\sum_{i=\tau(j)+1}^{n} (\theta_{i-1} Z_i - \mu_j) \leq -1 \text{ for some } n \in (\tau(j),\ \tau(j) + 2\lceil \mu_j^{-1} \rceil]\right) \leq (C_4 + C_5)\, a_j^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}.$$

On this last event, $\tau(j+1) \le \tau(j) + 2\lceil \mu_j^{-1} \rceil$, and so (since $\mu_j > 0$),

$$P\left(\sum_{i=\tau(j)+1}^{n} \theta_{i-1} Z_i \le -1 \text{ for some } n \in (\tau(j),\ \tau(j+1)] \right) \le (C_4 + C_5)\, a_j^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})} .$$

Summing over $j' \ge j$, it follows that

$$P\left(M_n \le j-1 \text{ for some } n > \tau(j)\right) \le C_6\, a_j^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}$$

for an appropriate constant C_6. This will imply $M_n \to \infty$ a.s. once we know $\tau(j) < \infty$ a.s. for all j. But, M_n executes a mean 0 random walk over $[\tau(j),\, \tau(j+1)]$, since θ_i is constant there. Such a random walk is recurrent and so, in fact, $\tau(j) < \infty$; hence, $M_n \to \infty$ a.s., as desired. ∎

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