

STA357 Background Facts, Fall 2026

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This document provides a few background mathematics and probability facts which will be used in STA357. (Note that students are responsible for knowing all needed background, at the level of STA257 or Chapters 1-4 of the Evans-Rosenthal book.)

This document might be updated throughout the semester.

Continuity of Probabilities [Rig §3.3, Evans §1.6]

Write $\{A_n\} \nearrow A$ to mean $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $\bigcup_n A_n = A$. (“ A_n increases to A ”)

Write $\{A_n\} \searrow A$ to mean $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ and $\bigcap_n A_n = A$. (“ A_n decreases to A ”)

Note that $\{A_n\} \searrow A \iff \{A_n^C\} \nearrow A^C$.

Fact: If $\{A_n\} \nearrow A$ or $\{A_n\} \searrow A$, then $\lim_{n \rightarrow \infty} \mathbf{P}(A_n) = \mathbf{P}(A)$.

Proof: Suppose $\{A_n\} \nearrow A$. Let $B_n = A_n \cap A_{n-1}^C$. Then the $\{B_n\}$ are disjoint, with $\bigcup B_n = \bigcup A_n = A$. Hence,

$$\mathbf{P}(A) = \mathbf{P}\left(\bigcup_{m=1}^{\infty} B_m\right) = \sum_{m=1}^{\infty} \mathbf{P}(B_m) = \lim_{n \rightarrow \infty} \sum_{m=1}^n \mathbf{P}(B_m) = \lim_{n \rightarrow \infty} \mathbf{P}\left(\bigcup_{m \leq n} B_m\right) = \lim_{n \rightarrow \infty} \mathbf{P}(A_n).$$

If instead $\{A_n\} \searrow A$, then $\{A_n^C\} \nearrow A^C$, so

$$\mathbf{P}(A) = 1 - \mathbf{P}(A^C) = 1 - \lim_{n \rightarrow \infty} \mathbf{P}(A_n^C) = \lim_{n \rightarrow \infty} (1 - \mathbf{P}(A_n^C)) = \lim_{n \rightarrow \infty} \mathbf{P}(A_n). \quad \blacksquare$$

Liminf and Limsup [Rig §A.4]

For a set of real numbers, their infimum is their largest lower bound (which equals their minimum if it exists), and their supremum is their smallest upper bound (which equals their maximum if it exists).

Then, for a sequence $x_1, x_2, \dots$ of real numbers, we define:

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} x_k \right) \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sup_{k \geq n} x_k \right).$$

These limits always exist (though they may be $\pm\infty$).

Furthermore, $\lim_{n \rightarrow \infty} x_n$ exists if and only if $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$, in which case

$$\lim_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n.$$

de Morgan's Laws and Set Theory [e.g. Rig §A.1]

We always have $(\bigcup_{\alpha} A_{\alpha})^C = \bigcap_{\alpha} A_{\alpha}^C$ and $(\bigcap_{\alpha} A_{\alpha})^C = \bigcup_{\alpha} A_{\alpha}^C$.

That is, complement converts unions to intersections, and vice-versa.

Also, we can interpret $\bigcap_{\alpha}$ as “for all α ”, and $\bigcup_{\alpha}$ as “there exists α ”.

For example, “For all $n \in \mathbf{N}$, there is $k \geq n$ such that A_n occurs” is equivalent to:

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n.$$

And, “For all $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $x_n < \epsilon$ for all $n \geq N$ ” is equivalent to:

$$\bigcap_{\epsilon > 0} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x_n < \epsilon\}.$$

Exponential Bound [e.g. Rig p35]

Fact: For any real number x (even negative), we always have $e^x \geq 1 + x$.

Proof: Let $f(x) = e^x - 1 - x$.

Then $f(0) = 0$, and $f'(0) = 0$. And, $f''(x) = e^x > 0$ for all x .

For $u > 0$, $f'(u) = f'(u) - f'(0) = \int_0^u f''(v) dv > 0$. Hence, for $x > 0$, $f(x) = \int_0^x f'(u) du > 0$.

For $u < 0$, $-f'(u) = f'(0) - f'(u) = \int_u^0 f''(v) dv > 0$, so $f'(u) < 0$. Hence, for $x < 0$, $-f(x) = f(0) - f(x) = \int_x^0 f'(u) du < 0$, so $f(x) > 0$.

Hence, in all cases, $f(x) \geq 0$, i.e. $e^x \geq 1 + x$. ■

Rational Replacement [e.g. Rig p59]

Consider a union such as $\left\{ \bigcup_{\epsilon > 0} |Z_n - Z| \geq \epsilon \text{ i.o.} \right\}$.

Is it equivalent to only take the union over rational $\epsilon > 0$? Yes!

Why? Well, clearly $\left\{ \bigcup_{\epsilon > 0} |Z_n - Z| \geq \epsilon \text{ i.o.} \right\} \supseteq \mathbf{P} \left(\bigcup_{\substack{\epsilon' > 0 \\ \epsilon' \text{ rational}}} |Z_n - Z| \geq \epsilon' \text{ i.o.} \right)$.

But given any $\epsilon > 0$, there exists a rational $\epsilon' \in (0, \epsilon)$. For this ϵ' , we have that

$$\{|Z_n - Z| \geq \epsilon \text{ i.o.}\} \subseteq \{|Z_n - Z| \geq \epsilon' \text{ i.o.}\}.$$

It follows that $\bigcup_{\epsilon > 0} \{|Z_n - Z| \geq \epsilon \text{ i.o.}\} \subseteq \bigcup_{\substack{\epsilon' > 0 \\ \epsilon' \text{ rational}}} \{|Z_n - Z| \geq \epsilon' \text{ i.o.}\}.$

Hence, we actually have $\bigcup_{\epsilon > 0} \{|Z_n - Z| \geq \epsilon \text{ i.o.}\} = \bigcup_{\substack{\epsilon' > 0 \\ \epsilon' \text{ rational}}} \{|Z_n - Z| \geq \epsilon' \text{ i.o.}\}.$ ■

Generalised Triangle Inequality [e.g. Rig p44]

For any real number r , clearly $-|r| \leq r \leq |r|$.

Hence, for a random variable X , $-|X| \leq X \leq |X|$, i.e. $-|X(s)| \leq X(s) \leq |X(s)|$.

So, by monotonicity, $-\mathbf{E}(|X|) \leq \mathbf{E}(X) \leq \mathbf{E}(|X|)$.

Hence, $|\mathbf{E}(X)| \leq \mathbf{E}(|X|)$. This is sometimes called the generalised triangle inequality.

If e.g. $X = a$ or $X = b$ with prob $\frac{1}{2}$ each, then this becomes $|\frac{1}{2}a + \frac{1}{2}b| \leq \frac{1}{2}|a| + \frac{1}{2}|b|$, i.e. $|a + b| \leq |a| + |b|$, which is the classic triangle inequality.